The explicit formula for the Hodrick-Prescott filter*

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Abstract

We obtain the exact analytical expression for the weights in the Hodrick-Prescott (HP) filter. We then use the expression for the weights to build a fast algorithm with computational improvements by a factor of up to fifty times in samples typical in macroeconomics. Our expression for the weights gives insights about the properties of the HP filter and we use it to propose an end-point bias correction of the filter. We illustrate the bias correction on the estimation of the National Bureau of Economic Research (NBER) recession dates and find that our estimates are closer to the NBER dates when compared to the usual HP filter estimates. Finally, we show that our derivations for the weights provide a methodology for finding the exact weights of the more general Whittaker-Henderson filters of which the HP filter is a particular case.

Keywords: Hodrick-Prescott filter; Whittaker-Henderson filter; exact weights; end-point bias; trend; cyclical component; turning point; smoothing parameter; Sherman-Morrison.

1 Introduction

In the past few decades, there has been an increasing interest among economists in techniques for detrending data and for representing their underlying trends. Without any consensus about which model represents the trend best, a popular alternative to model-based detrending is to use smoothing filters. Probably the filter that raised the most interest in economics is the Hodrick-Prescott filter (Hodrick, R. and Prescott, E. (1997)). The HP filter has, for a long period, been central for business cycle research; see King, R. G. and Rebelo, S. T. (1999) survey paper. Despite its widespread

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use[1], the explicit formulae for the weights of the HP filter have not been previously obtained. The lack of explicit formulae for the weights limits the understanding of the characteristics of the detrended data, of the end-points estimates of the trend and of the choice of the smoothing parameter; see Baxter, M. and King, R. G. (1999); King, R. G. and Rebelo, S. T. (1993). Moreover, the HP filter can generate spurious cyles/correlations in the filtered data with negative impact on estimation and inference; see Christiano, L. and den Haan, W. (1996); Cogley, T. and Nason, J. M. (1995); Harvey, A. C. and Jaeger, A. (1993); Singleton, K.J. (1988). Knowledge of the weights in the HP filter can help alleviate these problems.

In this paper we address several gaps in the literature. We first derive the explicit formulae for the weights of the HP filter. We then use the formulae to propose a solution for reducing the bias of the trend estimates at the end of the sample. Using U.S. real GDP from 1947:Q1 to 2013:Q2, our solution provides estimates of the recession dates which are identical or much closer to the NBER recession dates, when compared with the usual estimates of the HP filter. In addition, we develop an algorithm for implementing the filter, which is up to fifty times faster with sample sizes typical in economics. Finally, we show that the rationale of our derivations for the weights of the HP filter, provide a methodology to obtain the weights of the more general Whittaker-Henderson filters.[2] Other than the contributions mentioned here, our formulae can also be used to derive analytically the moments needed in the estimation of dynamic stochastic general equilibrium (DSGE) models in order to reduce the computation time (Gorodnichenko, Y. and Ng, S. (2010)), and to propose a solution for spurious cycles and moments by reducing the end-point bias or the fat-tailedness/skewness of statistics of interest (Christiano, L. and den Haan, W. (1996)).

Given a vector of time series $\boldsymbol{y} = (y_1, \ldots, y_n)'$, the HP filter decomposes each y_i into a trend component τ_i and a cyclical component c_i, *i.e.* $y_i := \tau_i + c_i$, $i = 1, \ldots, n$. The trend component is estimated as $\hat{\boldsymbol{\tau}} = (\hat{\tau}_1, \ldots, \hat{\tau}_n)'$ through the solution of the constrained minimization problem

$$\min_{\tau_1,\cdots,\tau_n} \sum_{i=1}^{n} (y_i - \tau_i)^2 + \alpha \sum_{i=2}^{n-1} (\tau_{i+1} - 2\tau_i + \tau_{i-1})^2 , \tag{1.1}$$

where α is a positive smoothing parameter that penalizes variability in the trend

[1] Recent articles that apply the HP filter include Angeloni, I. and Faia, E. (2013); Bai, Y. and Zhang, J. (2010); Bengui, J., Mendoza, E. G. and Quadrini, V. (2013); Boldea, O. and Hall, A. R. (2013); Broner, F., Didier, T., Erce, A. and Schmukler, S. L. (2013); Champagne, J. and Kurmann, A. (2013); Coibion, O. and Gorodnichenko, Y. (2011); Corradi, V., Distaso, W. and Mele, A. (2013); Donaldson, J. B., Gershun, N. and Giannoni, M. P. (2013); Duca, M. L. and Peltonen, T. A. (2013); Edmond, C. and Weill, P.-O. (2012); Elsby, M. W and Shapiro, M. D. (2012); Lu, S.-S. (2013); Lugauer S. (2012); Hansen, P. R., Lunde, A. and Nason, J. M. (2011); Gospodinov, N. and Ng, S. (2013); Kryvtsov, O. and Midrigan, V. (2013); Markiewicz, A. (2012); Mandelman, F. S. and Zlate, A. (2012); Moscarini, G. and Postel-Vinay, F. (2012); Ohanian, L. E. and Raffo, A. (2012); Ramadorai, T. (2012); Ravn, M. O., Schmitt-Grohé, S. and Uribe, M. (2012); Robin, J.-M. (2011); Tsyrennikov, V. (2013); Schmitt-Grohé, S. and Uribe, M. (2012).

[2] Whittaker, E. T. (1923) was the first who suggested the idea of smoothing filters and there has been a large subsequent literature on filters of this type; see Kitagawa, G. and Gersch , W. (1996).

component.

The unique solution to the minimization problem in (1.1) is

$$\hat{\boldsymbol{\tau}} = (\boldsymbol{I}_n + \alpha \boldsymbol{F})^{-1} \boldsymbol{y}, \tag{1.2}$$

where $\boldsymbol{I}_n$ is an $n \times n$ identity matrix and $\boldsymbol{F}$ is the pentadiagonal $n \times n$ matrix

$$\boldsymbol{F} := \begin{pmatrix} 1 & -2 & 1 & & & & \\ -2 & 5 & -4 & \ddots & & & \\ 1 & -4 & 6 & \ddots & \ddots & & \\ & \ddots & \ddots & \ddots & \ddots & \ddots & \\ & & \ddots & \ddots & 6 & -4 & 1 \\ & & & \ddots & -4 & 5 & -2 \\ & & & & 1 & -2 & 1 \end{pmatrix},$$

$n \geq 5$, where undisplayed elements are zero. The larger is the value of α, the smoother $\hat{\boldsymbol{\tau}}$ is. If $\alpha = 0$, then $\hat{\boldsymbol{\tau}} = \boldsymbol{y}$. If $\alpha \to \infty$, then $\hat{\boldsymbol{\tau}}$ is the ordinary least square estimate of $\boldsymbol{\tau}$. Both the trend component and the cyclical component estimates are weighted averages of the y's, *i.e.*

$$\hat{\tau}_i = \sum_{j=1}^{n} p_{i,j} y_j, \quad \hat{c}_i = y_i - \hat{\tau}_i, \quad i = 1, \ldots, n. \tag{1.3}$$

Knowledge of the formulae for the weights $p_{i,j}$ is important in understanding the properties of the trend/cyclical component; in developing methods for estimating α; in solving the end-of-sample bias of the filter; in implementing more easily the filter in software; in reducing computational time; in computing the moments of the HP filtered model variables; etc. As previously mentioned, the exact formulae for $p_{i,j}$ have not been previously derived in the literature. Approximations to it were obtained, for example, by McElroy, T. (2008) who worked with the spectral factorization of the HP filter. The main drawback of McElroy's result is that his formulae provide good approximations to $p_{i,j}$ away from the end-points of the sample and only for large n. More recently, De Jong, R. M. and Sakarya, N. (2013) have derived a new representation for $p_{i,j}$ using basis functions. In this paper we obtain the *exact* formulae for $p_{i,j}$. Our formulae are not only exact, but are also simpler. This is the first contribution of our paper.

The second contribution of our paper is to show that $(\boldsymbol{I}_n + \alpha \boldsymbol{F})^{-1}$ can be computed using only a few matrices of size $m \times m$, where $m := \lfloor n/2 \rfloor$ is the least integer of $n/2$. This can reduce the computational time by a factor of up to fifty times. Recently, economists have had access to data at higher frequencies (daily and intraday data), hence they have to analyze large samples; see Adrian, T. and Rosenberg, J. (2008); Harris, R. D. F., Evarist, S. and Fatih, Y. (2011), etc. However, even with data at smaller frequencies (monthly, quarterly), the estimation of DSGE models has to rely on the computation of the moments of the HP filtered data and HP filtered model variables over a grid of values for the parameters in the model, which is time consuming; see Gorodnichenko, Y. and Ng, S. (2010). As a consequence, processing the data in timely manner without exceeding memory capacity is still challenging despite the continuing progress in computer technology. To meet this challenge we fully exploit the mathematical structure of the weights in order to develop an algorithm for the HP filter that can be implemented in software. Our results can also be applied in other areas such as communications, surveillance applications, network traffic, where it is known that sample sizes are very large.

The third contribution of this paper is a solution to the end-point bias of the HP filter using our formulae for the weights. Because not all the trend components in (1.1) are treated equally in the minimization problem, the HP filter is less efficient at the beginning and at the end of the sample; see Baxter, M. and King, R. G. (1999); Christiano, L. and den Haan, W. (1996); Kaiser, R. and Maravall, A. (1999); King, R. G. and Rebelo, S. T. (1993); Mise, E., Kim, T.-H. and Newbold, P. (2005). We illustrate our solution on the estimation of the turning points dated by NBER. Our solution can be adapted to other type of applications, such as the reduction of the end-point bias in the computation of the moments in the generalized method of moments (GMM) estimation and related statistics; see Christiano, L. and den Haan, W. (1996).

The final contribution of our paper is to show that the computations for deriving the exact formulae for $p_{i,j}$, provide a methodology for computing the weights in more general filters where the trend component is estimated as the solution of the following minimization problems

$$\min_{\tau_1,\cdots,\tau_n} \sum_{i=1}^{n}(y_i-\tau_i)^2+\alpha\sum_{i=1}^{n-r}\left(\Delta^r\tau_j\right)^2, (1.4)$$

and

$$\min_{\tau_1,\cdots,\tau_n} \sum_{i=1}^{n}(y_i-\tau_i)^2+\alpha_1\sum_{i=1}^{n-1}\left(\Delta\tau_j\right)^2+\cdots+\alpha_r\sum_{i=1}^{n-r}\left(\Delta^r\tau_j\right)^2, \tag{1.5}$$

where $\alpha, \alpha_1, \ldots, \alpha_r$ are positive smoothing parameters and Δ is the forward difference operator: $\Delta\tau_j = \tau_{j+1} - \tau_j$. These filters are known as Whittaker-Henderson filters and have been proposed by Whittaker, E. T. (1923) and Henderson, R. (1924) for applications in actuarial finance. The filter in (1.5) can be seen as a discrete spline smoothing where $\alpha_1, \ldots, \alpha_r$ correspond to a specification of the choice of number and location of knots in the spline-fitting process. The HP filter is a special case of (1.4) with $r = 2$, where $\Delta^2\tau_j = \tau_{j+2} - 2\tau_{j-1} + \tau_j$, and was popularized in economics by Hodrick, R. and Prescott, E. (1997).

This paper is organized as follows. In Section 2, we derive the formula for the exact weights in the HP filter. In Section 3 we introduce the results about the reduction in the computation time of the HP filter and illustrate them in a Monte

Carlo simulation study. In Section 4, the solution to the end of sample bias is proposed and its performance is illustrated with the estimation of the turning points in U.S. Section 5 generalises the derivation of the HP filter weights to the filters (1.4) and (1.5). Section 6 concludes. The proofs are relegated to the Appendix. We use the notation in Abadir, K. M. and Magnus, J. R. (2002). Links to the Matlab programs containing our results are provided in each section.

2 The exact weights of the HP filter

In order to obtain the exact inverse of $\boldsymbol{I}_n + \alpha \boldsymbol{F}$, note that

$$\boldsymbol{F} = \boldsymbol{Q}\boldsymbol{Q} - \boldsymbol{g}\boldsymbol{g}' - \boldsymbol{P}_n \boldsymbol{g}\boldsymbol{g}' \boldsymbol{P}_n, \tag{2.1}$$

where $\boldsymbol{Q}$ is a tridiagonal matrix of size $n \times n$,

$$\boldsymbol{Q} := \begin{pmatrix} 2 & -1 & & & & & \\ -1 & 2 & -1 & & & & \\ & -1 & 2 & -1 & & & \\ & & \ddots & \ddots & \ddots & & \\ & & & -1 & 2 & -1 & \\ & & & & -1 & 2 & -1 \\ & & & & & -1 & 2 \end{pmatrix}, \tag{2.2}$$

$\boldsymbol{P}_n$ is a permutation matrix of size $n \times n$,

$$\boldsymbol{P}_n := \begin{pmatrix} & & & 1 \\ & & \cdot^{\cdot^{\cdot}} & \\ & \cdot^{\cdot^{\cdot}} & & \\ 1 & & & \end{pmatrix}, \tag{2.3}$$

and $\boldsymbol{g} := (-2, 1, \boldsymbol{0}')'$. The pentadiagonal matrix $\boldsymbol{Q}\boldsymbol{Q}$ has full rank, while $\boldsymbol{g}\boldsymbol{g}'$ and $\boldsymbol{P}_n \boldsymbol{g}\boldsymbol{g}' \boldsymbol{P}_n$ have rank 1 which allows us to obtain a simple expression for $(\boldsymbol{I}_n + \alpha \boldsymbol{F})^{-1}$ by applying the Sherman-Morrison formula (Abadir, K. M. and Magnus, J. R. (2005), p.248) twice.

Note that $\boldsymbol{Q}$ has distinct eigenvalues

$$\gamma_j = 2 - 2\cos\left(\frac{\pi j}{n+1}\right), \quad j = 1, \ldots, n, \tag{2.4}$$

and corresponding eigenvector $\boldsymbol{x}_j = (x_{1j}, \ldots, x_{nj})'$ with

$$x_{i,j} = \left(\frac{2}{n+1}\right)^{1/2} \sin\left(\frac{\pi i j}{n+1}\right), \quad i, j = 1, \ldots, n. \tag{2.5}$$

Theorem 2.1 below gives the exact inverse of $\boldsymbol{I}_n + \alpha \boldsymbol{F}$ in terms of only α, n and the eigenvalues/eigenvectors of $\boldsymbol{Q}$. We denote by $\boldsymbol{T}$ the $n \times n$ matrix of eigenvectors of $\boldsymbol{Q}$ with typical element $x_{i,j}$. Also let

$$\boldsymbol{\Lambda} := \mathrm{diag}\left(\lambda_1, \ldots, \lambda_n\right), \tag{2.6}$$

with

$$\lambda_j = 1 + \alpha \gamma_j^2, \tag{2.7}$$

the eigenvalues of $\boldsymbol{I}_n + \alpha \boldsymbol{Q}\boldsymbol{Q}$, $j = 1, \ldots, n$. Denote by k_1 and k_2 two scalars defined as

$$k_1 := \frac{2\alpha}{1 - 2\alpha \sum\limits_{j=1,\mathrm{odd}}^{n} \left(2x_{1,j} - x_{2,j}\right)^2 \lambda_j^{-1}}, \tag{2.8}$$

$$k_2 := \frac{2\alpha}{1 - 2\alpha \sum\limits_{j=1,\mathrm{even}}^{n} \left(2x_{1,j} - x_{2,j}\right)^2 \lambda_j^{-1}}. \tag{2.9}$$

Finally let $\boldsymbol{K}_1$ and $\boldsymbol{K}_2$ denote two $n \times n$ matrices with typical element for row i and column j,

$$\frac{\left(2x_{i,1} - x_{i,2}\right)\left(2x_{1,j} - x_{2,j}\right)}{\lambda_i \lambda_j}, \quad i, j = 1, \ldots, n, \tag{2.10}$$

for $i + j$ even and j odd in $\boldsymbol{K}_1$, and $i + j$ even and j even in $\boldsymbol{K}_2$, the rest of the elements being zero. We are now in the position to give the following theorem.

Theorem 2.1.

$$\left(\boldsymbol{I}_n + \alpha \boldsymbol{F}\right)^{-1} = \boldsymbol{T}\boldsymbol{\Lambda}^{-1}\boldsymbol{T} + k_1 \boldsymbol{T}\boldsymbol{K}_1\boldsymbol{T} + k_2 \boldsymbol{T}\boldsymbol{K}_2\boldsymbol{T}, \tag{2.11}$$

where $\boldsymbol{\Lambda}^{-1} := \mathit{diag}\left(\lambda_1^{-1}, \ldots, \lambda_n^{-1}\right)$.

The proof is relegated to the Appendix. The link to the Matlab program for Theorem 2.1 is here: https://dl.dropboxusercontent.com/u/113649213/HPTheorem1vsOldHP.m.

Corollary 2.1. *Let* $\hat{\tau}_i = \sum_{j=1}^{n} p_{i,j} y_j$ *be the trend component estimate for observation* y_i, $i = 1, \ldots, n$. *The weights* $p_{i,j}$ *are given by*

$$p_{i,j} = \sum_{s=1}^{n} \frac{x_{i,s} x_{s,j}}{\lambda_s} \tag{2.12}$$

$$+ k_1 \sum_{t=1,odd}^{n} \sum_{s=1,odd}^{n} x_{i,s} \frac{(2x_{1,s} - x_{2,s})(2x_{1,t} - x_{2,t})}{\lambda_s \lambda_t} x_{t,j} \tag{2.13}$$

$$+ k_2 \sum_{t=1,even}^{n} \sum_{s=1,even}^{n} x_{i,s} \frac{(2x_{1,s} - x_{2,s})(2x_{1,t} - x_{2,t})}{\lambda_s \lambda_t} x_{t,j}, \tag{2.14}$$

where k_1 *and* k_2 *are as in (2.8) and (2.9).*

The proof follows by simply computing the matrix multiplications in Theorem 2.1. Since

$$x_{i,j} = x_{j,i}, \tag{2.15}$$

the matrices $\boldsymbol{T}$, $\boldsymbol{K}_1$ and $\boldsymbol{K}_2$ are symmetric. Hence $(\boldsymbol{I}_n + \alpha \boldsymbol{F})^{-1}$ is also symmetric. Also, note that

$$x_{i,j} = (-1)^{j-1} x_{n+1-i,j} \tag{2.16}$$

which together with (2.15) imply that

$$x_{i,j} = (-1)^{i-1} x_{i,n+1-j}, \quad i, j = 1, \dots, n. \tag{2.17}$$

By (2.16) and (2.17) we have that $x_{i,s} x_{s,j} = x_{n+1-i,s} x_{s,n+1-j}$, $s = 1, \dots, n$. Hence

$$p_{i,j} = p_{n+1-i,n+1-j}, \tag{2.18}$$

property which indicates that $\boldsymbol{T}\boldsymbol{\Lambda}^{-1}\boldsymbol{T}$, $\boldsymbol{T}\boldsymbol{K}_1\boldsymbol{T}$, $\boldsymbol{T}\boldsymbol{K}_2\boldsymbol{T}$ and $(\boldsymbol{I}_n + \alpha \boldsymbol{F})^{-1}$ are centrosymmetric (symmetric about their center) and hence bisymmetric (symmetric about the main diagonals).

Moreover, for large n and away from the end points of the sample, we have the following corollary for the terms in $p_{i,j}$.

Corollary 2.2. *Pointwise in $i > 0$, $j > 0$ and $\alpha > 0$, as $n \to \infty$,*
(a) the limit of the constants in (2.8) and (2.9) is

$$\lim_{n\to\infty} k_1 = \lim_{n\to\infty} k_2 := k, \tag{2.19}$$

where

$$k := 2\alpha \left(1 - 4\alpha \int_0^1 \frac{16\,(\sin(r\pi))^4\,(\sin(2r\pi))^2}{1 + 16\alpha\,(\sin(r\pi))^4}\,\mathrm{d}r \right)^{-1}; \tag{2.20}$$

(b) the limit of the term in (2.12) is

$$\lim_{n\to\infty} \sum_{s=1}^{n} \frac{x_{i,s} x_{s,j}}{\lambda_s} = 2 \int_0^1 \frac{\sin(ir\pi)\sin(jr\pi)}{1 + 16\alpha\,(\sin(r\pi))^4}\,\mathrm{d}r; \tag{2.21}$$

(c) the limit of the terms in (2.13) and (2.14) is

$$\begin{aligned} &\lim_{n\to\infty} \sum_{t=1,odd}^{n} \sum_{s=1,odd}^{n} x_{i,s} \frac{(2x_{1,s} - x_{2,s})(2x_{1,t} - x_{2,t})}{\lambda_s \lambda_t} x_{t,j} \qquad (2.22) \\ = \;& \lim_{n\to\infty} \sum_{t=1,even}^{n} \sum_{s=1,even}^{n} x_{i,s} \frac{(2x_{1,s} - x_{2,s})(2x_{1,t} - x_{2,t})}{\lambda_s \lambda_t} x_{t,j} \\ = \;& 1024 \int_0^1 \int_0^1 \frac{\sin(2ir\pi)\,(\sin(r\pi))^4}{\left(1 + 16\alpha\,(\sin(r\pi))^4\right)} \\ & \times \frac{(\sin(u\pi))^4 \sin(2ju\pi)}{\left(1 + 16\alpha\,(\sin(u\pi))^4\right)}\,\mathrm{d}r\,\mathrm{d}u. \end{aligned}$$

See the Appendix for the proof.

The matrix $\boldsymbol{K}_1$ ($\boldsymbol{K}_2$) has the odd (even) rows and columns equal to zero. The nonzero elements of these matrices are weighted by k_1 and k_2 which are identical only for $n \to \infty$, as it can be seen from (2.19). Furthermore, the second term in (2.20) converges to a constant as $\alpha \to \infty$,

$$\lim_{\alpha\to\infty} \alpha \int_0^1 \frac{16\left(\sin(r\pi)\right)^4\left(\sin(2r\pi)\right)^2}{1+16\alpha\left(\sin(r\pi)\right)^4}\,\mathrm{d}r = \int_0^1 \left(\sin(2r\pi)\right)^2\mathrm{d}r = \frac{1}{2}. \tag{2.23}$$

Hence, from (2.20) and (2.23) it follows that

$$\lim_{\alpha\to\infty}\lim_{n\to\infty} k_1 = \lim_{\alpha\to\infty}\lim_{n\to\infty} k_2 = \infty. \tag{2.24}$$

Also, as $\alpha \to \infty$, the limit of (2.21) is

$$\lim_{\alpha\to\infty} \int_0^1 \frac{\sin(ir\pi)\sin(jr\pi)}{1+16\alpha\left(\sin(r\pi)\right)^4}\,\mathrm{d}r = 0, \tag{2.25}$$

pointwise in $i > 0$ and $j > 0$, away from the end-points of the sample. Moreover, away from the end-points of the sample, by l'Hôpital's rule, (2.22) converges to zero as α and $n \to \infty$. Thus, as n and α become larger, the weights become smaller.

3 Reducing the computation time in the HP filter

Theorem 2.1 and Corollary 2.1 allow us to greatly reduce the computation time of the weights in the HP filter by working with matrices of size $m\times m$, where $m := \lfloor n/2 \rfloor$ is the least integer of $n/2$, instead of matrices of size $n \times n$. To illustrate this we denote by $\boldsymbol{P}_m$ a similar permutation matrix to $\boldsymbol{P}_n$ given in (2.3), but of size $m \times m$, and give the following corollaries.

Corollary 3.1. *The matrix $\boldsymbol{T}\boldsymbol{\Lambda}^{-1}\boldsymbol{T}$ from (2.11) can be written as*

$$\boldsymbol{T}\boldsymbol{\Lambda}^{-1}\boldsymbol{T} = \begin{pmatrix} \boldsymbol{V}_1 & \boldsymbol{V}_2 \\ \boldsymbol{P}_m\boldsymbol{V}_2\boldsymbol{P}_m & \boldsymbol{P}_m\boldsymbol{V}_1\boldsymbol{P}_m \end{pmatrix}, \text{ for } n \text{ even}, \tag{3.1}$$

$$\boldsymbol{T}\boldsymbol{\Lambda}^{-1}\boldsymbol{T} = \begin{pmatrix} \boldsymbol{V}_1 & \boldsymbol{v} & \boldsymbol{V}_2 \\ \boldsymbol{v}' & v_{m+1,m+1} & \boldsymbol{v}'\boldsymbol{P}_m \\ \boldsymbol{P}_m\boldsymbol{V}_2\boldsymbol{P}_m & \boldsymbol{P}_m\boldsymbol{v} & \boldsymbol{P}_m\boldsymbol{V}_1\boldsymbol{P}_m \end{pmatrix}, \text{ for } n \text{ odd}, \tag{3.2}$$

where $\boldsymbol{V}_1$ is a $m\times m$ matrix with typical element given by (2.12), $i,j = 1,\ldots,m$; $\boldsymbol{V}_2$ is a $m\times m$ matrix with typical element given by (2.12), $i = 1,\ldots,m$ and $j = m+1,\ldots n$ if n is even or $j = m+2,\ldots,n$ if n is odd; $\boldsymbol{v}$ is a column vector of length m with typical element as in (2.12) with $i = 1,\ldots,m$ and $j = m+1$; $v_{m+1,m+1}$ is given by (2.12) where $i,j = m+1$.

The proof follows from Weaver, J. R. (1985), the corollary being a simple consequence of the fact that $\boldsymbol{T}\boldsymbol{\Lambda}^{-1}\boldsymbol{T}$ is centrosymmetric.

Corollary 3.2. *The matrix* $\boldsymbol{T K}_1 \boldsymbol{T}$ *from (2.11) can be written as*

$$\boldsymbol{T K}_1 \boldsymbol{T} = \begin{pmatrix} \boldsymbol{D} & \boldsymbol{D P}_m \\ \boldsymbol{P}_m \boldsymbol{D}' & \boldsymbol{P}_m \boldsymbol{D P}_m \end{pmatrix}, \text{ for } n \text{ even,} \tag{3.3}$$

$$\boldsymbol{T K}_1 \boldsymbol{T} = \begin{pmatrix} \boldsymbol{D} & \boldsymbol{d} & \boldsymbol{D P}_m \\ \boldsymbol{d}' & d_{m+1,m+1} & \boldsymbol{d}' \boldsymbol{P}_m \\ \boldsymbol{P}_m \boldsymbol{D}' & \boldsymbol{P}_m \boldsymbol{d} & \boldsymbol{P}_m \boldsymbol{D P}_m \end{pmatrix}, \text{ for } n \text{ odd,} \tag{3.4}$$

where $\boldsymbol{D}$ *is a* $m \times m$ *matrix with typical element given by (2.13),* $i, j = 1, \ldots, m$*;* $\boldsymbol{d}$ *is a column vector of length* m *with typical element as in (2.13), where* $i = 1, \ldots, m$ *and* $j = m + 1$*;* $d_{m+1,m+1}$ *is the term in (2.13) with* $i, j = m + 1$*.*

The proof of this corollary follows from Weaver, J. R. (1985) and is a consequence of the fact that $\boldsymbol{T K}_1 \boldsymbol{T}$ is centrosymmetric. In the upper-right corners of (3.3) and (3.4) we have a permutation of $\boldsymbol{D}$. This follows from (2.16) and (2.17) by noticing that for s odd, $x_{j,s} = x_{s,n+1-j}$. As a consequence, in (2.13) when $\hat{\tau}_i$ is computed, y_j and y_{n+j-1} receive the same weight, $i, j = 1, \ldots, n$.

Corollary 3.3. *The matrix* $\boldsymbol{T K}_2 \boldsymbol{T}$ *from (2.11) can be written as*

$$\boldsymbol{T K}_2 \boldsymbol{T} = \begin{pmatrix} \boldsymbol{E} & -\boldsymbol{E P}_m \\ -\boldsymbol{P}_m \boldsymbol{E}' & \boldsymbol{P}_m \boldsymbol{E P}_m \end{pmatrix}, \text{ for } n \text{ even,} \tag{3.5}$$

$$\boldsymbol{T K}_2 \boldsymbol{T} = \begin{pmatrix} \boldsymbol{E} & \boldsymbol{e} & -\boldsymbol{E P}_m \\ \boldsymbol{e}' & e_{m+1,m+1} & -\boldsymbol{e}' \boldsymbol{P}_m \\ -\boldsymbol{P}_m \boldsymbol{E}' & -\boldsymbol{P}_m \boldsymbol{e} & \boldsymbol{P}_m \boldsymbol{E P}_m \end{pmatrix}, \text{ for } n \text{ odd,} \tag{3.6}$$

where $\boldsymbol{E}$ *is a* $m \times m$ *matrix with typical element given by (2.14),* $i, j = 1, \ldots, m$*;* $\boldsymbol{e}$ *is a column vector of length* m *with typical element as in (2.14), where* $i = 1, \ldots, m$ *and* $j = m + 1$*;* $e_{m+1,m+1}$ *is the term in (2.14) with* $i, j = m + 1$*.*

The proof follows from Weaver, J. R. (1985) and is a consequence of the fact that $\boldsymbol{T K}_2 \boldsymbol{T}$ is centrosymmetric. In the upper-right corners of (3.5) and (3.6) we have a permutation of $-\boldsymbol{E}$. This follows from (2.16) and (2.17) by noticing that for s even, $x_{j,s} = -x_{s,n+1-j}$. As a consequence, in (2.14) when $\hat{\tau}_i$ is computed, y_j and y_{n+j-1} receive the same weight, but of opposite sign, $i, j = 1, \ldots, n$.

An important consequence of Corollaries 3.1, 3.2 and 3.3 is the following simplification of Theorem 2.1.

Corollary 3.4. *For* n *even,*

$$(\boldsymbol{I}_n + \alpha \boldsymbol{F})^{-1} = \begin{pmatrix} \boldsymbol{V}_1 + \boldsymbol{D} + \boldsymbol{E} & \boldsymbol{V}_2 + (\boldsymbol{D} - \boldsymbol{E}) \boldsymbol{P}_m \\ \boldsymbol{P}_m (\boldsymbol{V}_2 \boldsymbol{P}_m + \boldsymbol{D}' - \boldsymbol{E}') & \boldsymbol{P}_m (\boldsymbol{V}_1 + \boldsymbol{D} + \boldsymbol{E}) \boldsymbol{P}_m \end{pmatrix}, \tag{3.7}$$

and for n *odd,*

$$(\boldsymbol{I}_n + \alpha \boldsymbol{F})^{-1} = \begin{pmatrix} \boldsymbol{V}_1 + \boldsymbol{D} + \boldsymbol{E} & \boldsymbol{a} & \boldsymbol{V}_2 + (\boldsymbol{D} - \boldsymbol{E}) \boldsymbol{P}_m \\ \boldsymbol{a}' & a & \boldsymbol{z}' \boldsymbol{P}_m \\ \boldsymbol{P}_m (\boldsymbol{V}_2 \boldsymbol{P}_m + \boldsymbol{D}' - \boldsymbol{E}') & \boldsymbol{P}_m \boldsymbol{z} & \boldsymbol{P}_m (\boldsymbol{V}_1 + \boldsymbol{D} + \boldsymbol{E}) \boldsymbol{P}_m \end{pmatrix}, \tag{3.8}$$

where $\boldsymbol{a} := \boldsymbol{v} + \boldsymbol{d} + \boldsymbol{e}$*,* $\boldsymbol{z} := \boldsymbol{v} + \boldsymbol{d} - \boldsymbol{e}$*,* $a := v_{m+1,m+1} + d_{m+1,m+1} + e_{m+1,m+1}$*.*

Corollary 3.4 suggests that $(\boldsymbol{I}_n + \alpha \boldsymbol{F})^{-1}$ which is of size $n \times n$, can be computed using only the matrices $\boldsymbol{P}_m, \boldsymbol{V}_1, \boldsymbol{V}_2, \boldsymbol{D}, \boldsymbol{E}$ which are of (smaller) size $m \times m$. The formulae for computing these matrices are given in the next corollary where we use the following notation. We denote by $\odot$ the Hadamard product. Let $\boldsymbol{T}_1$ be a $m \times m$ matrix with typical element given in (2.5), but with $i, j = 1, \ldots, m$. Let $\boldsymbol{J}$ denote a $m \times m$ matrix given by $\boldsymbol{J} := (\boldsymbol{\imath}, -\boldsymbol{\imath}, \boldsymbol{\imath}, \ldots, \boldsymbol{\imath}, -\boldsymbol{\imath})$, where $\boldsymbol{\imath}$ is a column vector of ones of size $m \times 1$. Using (2.15), (2.16) and (2.17), we have an alternative representation of the matrix $\boldsymbol{T}$ in terms of a 2×2 block matrix for n even,

$$\boldsymbol{T} = \begin{pmatrix} \boldsymbol{T}_1 & (\boldsymbol{T}_1 \odot \boldsymbol{J})' \boldsymbol{P}_m \\ \boldsymbol{P}_m \boldsymbol{T}_1 \odot \boldsymbol{J} & (-1)^l \boldsymbol{P}_m (\boldsymbol{T}_1 \odot \boldsymbol{J})' \boldsymbol{P}_m \odot \boldsymbol{J} \end{pmatrix}, \tag{3.9}$$

and in terms of a 3×3 matrix for n odd,

$$\boldsymbol{T} = \begin{pmatrix} \boldsymbol{T}_1 & \boldsymbol{x}_1 & (\boldsymbol{T}_1 \odot \boldsymbol{J})' \boldsymbol{P}_m \\ \boldsymbol{x}_1' & x_{m+1,m+1} & \boldsymbol{x}_2' \\ \boldsymbol{P}_m \boldsymbol{T}_1 \odot \boldsymbol{J} & \boldsymbol{x}_2 & (-1)^l \boldsymbol{P}_m (\boldsymbol{T}_1 \odot \boldsymbol{J})' \boldsymbol{P}_m \odot \boldsymbol{J} \end{pmatrix}, \tag{3.10}$$

where $\boldsymbol{x}_1$ is a $m \times 1$ column vector with typical element given in (2.5) with $i = 1, \ldots, m$ and $j = m + 1$; $\boldsymbol{x}_2$ is a $m \times 1$ column vector with typical element given in (2.5) with $i = m + 2, m + 3 \ldots, 2m$ and $j = m + 1$; the scalar $x_{m+1,m+1}$ is computed as in (2.5) with $i, j = m + 1$ and

$$l := \begin{cases} 2, & \text{if } n = 4j \text{ or } n = 4j - 1, \text{ with } j \in \mathbb{N}, \\ 1, & \text{for the other values of } n. \end{cases} \tag{3.11}$$

Note that $\boldsymbol{T}$ is not centrosymmetric due to (2.16) and (2.17).

Let $\boldsymbol{b}$ denote the $m \times 1$ vector with typical element given by $\cos(\pi j/(n+1))$, $j = 1, \ldots, m$. Since $\cos(\pi j/(n+1)) = -\cos(\pi(n+1-j)/(n+1))$, then the eigenvalues of $\boldsymbol{I}_n + \alpha \boldsymbol{Q}\boldsymbol{Q}$ are given by the elements of the $n \times 1$ vector, for n even

$$\boldsymbol{\lambda} = \begin{pmatrix} \boldsymbol{\lambda}_1 \\ \boldsymbol{\lambda}_2 \end{pmatrix} := \begin{pmatrix} 1 + 4\alpha(1 - \boldsymbol{b}) \odot (1 - \boldsymbol{b}) \\ 1 + 4\alpha(1 + \boldsymbol{P}_m \boldsymbol{b}) \odot (1 + \boldsymbol{P}_m \boldsymbol{b}) \end{pmatrix}. \tag{3.12}$$

The matrix $\boldsymbol{\Lambda}$ from (2.6) can also be written in partitioned form

$$\boldsymbol{\Lambda} = \begin{pmatrix} \operatorname{diag}(\boldsymbol{\lambda}_1) & \mathbf{O}_{m,m} \\ \mathbf{O}_{m,m} & \operatorname{diag}(\boldsymbol{\lambda}_2) \end{pmatrix}, \text{ for } n \text{ even}, \tag{3.13}$$

$$\boldsymbol{\Lambda} = \begin{pmatrix} \operatorname{diag}(\boldsymbol{\lambda}_1) & \mathbf{0}_{m,1} & \mathbf{O}_{m,m} \\ \mathbf{0}_{1,m} & \lambda_{m+1} & \mathbf{0}_{1,m} \\ \mathbf{O}_{m,m} & \mathbf{0}_{m,1} & \operatorname{diag}(\boldsymbol{\lambda}_2) \end{pmatrix}, \text{ for } n \text{ odd}, \tag{3.14}$$

where λ_{m+1} is computed as in (2.7) with $j = m + 1$.

Let $\boldsymbol{G}_1$ be the $m \times m$ matrix with typical element for row s column t given by

$$\frac{(2x_{1,2s+1} - x_{2,2s+1})(2x_{1,2t+1} - x_{2,2t+1})}{\lambda_{2s+1}\lambda_{2t+1}}, \quad s, t = 0, \ldots, m - 1_{n \text{ even}}, \tag{3.15}$$

where $1_{n \text{ even}}$ is the indicator function which equals 1 if n is even, 0 if n odd. Let $\boldsymbol{G}_2$ be the $m \times m$ matrix with typical element for row s column t given by

$$\frac{(2x_{1,2s} - x_{2,2s})(2x_{1,2t} - x_{2,2t})}{\lambda_{2s}\lambda_{2t}}, \quad s, t = 1, \ldots, m. \tag{3.16}$$

Finally, let $\boldsymbol{M}_1$ be the $m \times m$ matrix with typical element $x_{i,2j+1}$, $i = 1, \ldots, m$, $j = 0, \ldots, m - 1_{n \text{ even}}$. Let $\boldsymbol{M}_2$ be the $m \times m$ matrix with typical element $x_{i,2j}$, $i, j = 1, \ldots, m$. We are now in the position to give the following corollary.

Corollary 3.5. *(a) The matrices $\boldsymbol{V}_1$, $\boldsymbol{V}_2$, $\boldsymbol{D}$, $\boldsymbol{E}$ from (3.1), (3.3), (3.5) are given by*

$$\boldsymbol{V}_1 := \begin{cases} \boldsymbol{W}_1, & n \text{ even}, \\ \boldsymbol{W}_1 + \boldsymbol{x}_1 \lambda_{m+1}^{-1} \boldsymbol{x}_1', & n \text{ odd}, \end{cases} \tag{3.17}$$

$$\boldsymbol{V}_2 := \begin{cases} \boldsymbol{W}_2, & n \text{ even}, \\ \boldsymbol{W}_2 + \boldsymbol{x}_1 \lambda_{m+1}^{-1} \boldsymbol{x}_2', & n \text{ odd}, \end{cases} \tag{3.18}$$

$$\boldsymbol{D} := \boldsymbol{M}_1 \boldsymbol{G}_1 \boldsymbol{M}_1', \tag{3.19}$$

$$\boldsymbol{E} := \boldsymbol{M}_2 \boldsymbol{G}_2 \boldsymbol{M}_2', \tag{3.20}$$

with

$$\boldsymbol{W}_1 := \boldsymbol{T}_1 \left(diag\left(\boldsymbol{\lambda}_1\right)\right)^{-1} \boldsymbol{T}_1 + \left(\boldsymbol{T}_1 \odot \boldsymbol{J}\right)' \boldsymbol{P}_m \left(diag\left(\boldsymbol{\lambda}_2\right)\right)^{-1} \boldsymbol{P}_m \boldsymbol{T}_1 \odot \boldsymbol{J}, \tag{3.21}$$

$$\begin{aligned} \boldsymbol{W}_2 := {} & \boldsymbol{T}_1 \left(diag\left(\boldsymbol{\lambda}_1\right)\right)^{-1} \left(\boldsymbol{T}_1 \odot \boldsymbol{J}\right)' \boldsymbol{P}_m \\ & + (-1)^l \left(\boldsymbol{T}_1 \odot \boldsymbol{J}\right)' \boldsymbol{P}_m \left(diag\left(\boldsymbol{\lambda}_2\right)\right)^{-1} \boldsymbol{P}_m \left(\boldsymbol{T}_1 \odot \boldsymbol{J}\right)' \boldsymbol{P}_m \odot \boldsymbol{J}, \end{aligned} \tag{3.22}$$

where l is defined in (3.11).

(b) For n odd, $\boldsymbol{v}$, $\boldsymbol{d}$ and $\boldsymbol{e}$ from (3.2), (3.4) and (3.6) are given by

$$\begin{aligned} \boldsymbol{v} := {} & \boldsymbol{T}_1 \left(diag\left(\boldsymbol{\lambda}_1\right)\right)^{-1} \boldsymbol{x}_1 + \boldsymbol{x}_1 \lambda_{m+1}^{-1} x_{m+1,m+1} \\ & + \left(\boldsymbol{T}_1 \odot \boldsymbol{J}\right)' \boldsymbol{P}_m \left(diag\left(\boldsymbol{\lambda}_2\right)\right)^{-1} \boldsymbol{x}_2. \end{aligned} \tag{3.23}$$

Let $i = 1, \ldots, m$ and $j = m + 1$, then $\boldsymbol{d}$ has typical element

$$\sum_{t=0}^{m} \sum_{s=0}^{m} x_{i,2s+1} \frac{\left(2x_{1,2s+1} - x_{2,2s+1}\right)\left(2x_{1,2t+1} - x_{2,2t+1}\right)}{\lambda_{2s+1}\lambda_{2t+1}} x_{2t+1,j}, \tag{3.24}$$

and $\boldsymbol{e}$ has typical element

$$\sum_{t=1}^{m} \sum_{s=1}^{m} x_{i,2s} \frac{\left(2x_{1,2s} - x_{2,2s}\right)\left(2x_{1,2t} - x_{2,2t}\right)}{\lambda_{2s}\lambda_{2t}} x_{2t,j}. \tag{3.25}$$

(c) The constants k_1 and k_2 from (2.8) and (2.9) are given by

$$k_1 := \frac{2\alpha}{1 - 2\alpha \sum_{j=0}^{m-1_{n \text{ even}}} \left(2x_{1,2j+1} - x_{2,2j+1}\right)^2 \lambda_{2j+1}^{-1}}, \tag{3.26}$$

$$k_2 := \frac{2\alpha}{1 - 2\alpha \sum_{j=1}^{m} \left(2x_{1,2j} - x_{2,2j}\right)^2 \lambda_{2j}^{-1}}. \tag{3.27}$$

The proof is omitted. It is a simple consequence of the fact that the expressions for $\boldsymbol{V}_1$ and $\boldsymbol{V}_2$ follow by multiplying by blocks the matrix $\boldsymbol{T}\boldsymbol{\Lambda}^{-1}\boldsymbol{T}$, where $\boldsymbol{T}$ and $\boldsymbol{\Lambda}$ are given in (3.9) and (3.13) for n even, and in (3.10) and (3.14) for n odd. When n is odd, k_1, $\boldsymbol{M}_1$ and $\boldsymbol{G}_1$ have to be computed accordingly, as indicated in (3.15) and (3.26). The link to the Matlab program for Corollaries 3.4 and 3.5 is here: https://dl.dropboxusercontent.com/u/113649213/OurHPvsOldHP.m.

3.1 Simulation study

We end this section with an illustrative simulation study meant to highlight the computational gains of Corollaries 3.4 and 3.5. We generate samples $\{y_i\}_{i=1}^{n}$ of sizes $n = 100, 500, 1000, 2000, 5000, 10000$, where $y_i = \tau_i + c_i$, with $\tau_i = \tau_{i-1} + u_i$, a random walk, $u_i \sim^{i.i.d.} N(0,1)$ and $c_i \sim^{i.i.d.} N(0,1)$. We avoid fixing the smoothing parameter to an arbitrary value (about which it seems to be no consensus; see Backus, D.K. and Kehoe, P. J. (1992); Baxter, M. and King, R. G. (1999); Giorno, C.L., Richardson, P., Roseveare, D. and van den Noord, P. (1995); Ravn, M. O. and Uhlig, H. (2002)). Instead we estimate it by the generalized cross validation (GCV) method introduced by Craven, P. and Wahba, G. (1979) for continuous spline smoothing. Brooks, R. J., Stone, M., Chan, F. Y. and Chan, L.K. (1988) have shown that GCV can also be used for the Whittaker-Henderson smoothing and implicitly for the HP filter. The estimate of the smoothing parameter is the one that minimizes the GCV criterion function

$$\min_{\alpha} n^{-1} \sum_{i=1}^{n} \left(\frac{y_i - \hat{\tau}_i}{1 - n^{-1}\text{trace}\left(\boldsymbol{I}_n + \alpha \boldsymbol{F}\right)^{-1}} \right)^2, \tag{3.28}$$

where, in our simulations, $\hat{\tau}_i$ and trace$(\boldsymbol{I}_n + \alpha\boldsymbol{F})^{-1}$ are computed for each value of $\alpha = 0.5, 1, 1.5, 2, \ldots, 19.5, 20$ (a grid of 40 values). Table 1 shows the computation time in seconds for the GCV method based on the formulae in Corollaries 3.4 and 3.5 with matrices of size $m \times m$ and also on the usual inversion formula for $(\boldsymbol{I}_n + \alpha\boldsymbol{F})^{-1}$ of size $n \times n$.

As it can be seen from Table 1, the difference in computation time between our formulae and the old formula for the HP filter are striking for a grid of only 40 values for α. For a data set comprising daily observations over two decades (as in Adrian, T. and Rosenberg, J. (2008) or in Harris, R. D. F., Evarist, S. and Fatih, Y. (2011)), the old HP formula takes 4626 seconds (one hour and half) to estimate α. The simulations were run on a very powerful computer: an Intel Xeon CPU 2.67 GHz, 24 GB RAM and 64-bit OS using Matlab R2012b. The Matlab programs used in this section can be found here: https://dl.dropboxusercontent.com/u/113649213/our HP.m and here: https://dl.dropboxusercontent.com/u/113649213/OldHP.m.

Table 1: Computation time in seconds for the GCV method for choosing α

n	Our HP	Old HP
100	0.05	2.25
500	0.81	3.18
1000	5.20	6.92
2000	30.99	104.16
5000	421.88	4626.31
10000	3217.30	40708.54

Even if an economist has data at lower frequency (monthly, quarterly), our formulae from Corollaries 3.4 and 3.5 can also be used to derive analytically the moments of the HP filtered model variables, which would help to reduce the computation time in the estimation of DSGE models; see Gorodnichenko, Y. and Ng, S. (2010).

4 A solution to the end-point bias with an application to the U.S. GDP

It is well known that the trend estimate of the HP filter for the most recent (and the first) observations is biased, as indicated for example in Baxter, M. and King, R. G. (1999); Mise, E., Kim, T.-H. and Newbold, P. (2005); Orphanides, A. and van Norden, S. (2002). As it can be seen from (1.1), the bias is due to the fact that not all the τ_i's are treated equally, resulting in higher weights $p_{i,j}$ at the beginning and end of the sample, $i = 1,\ 2,\ n-1,\ n$; $j = 1, \ldots, n$.

The solution proposed in the literature to solve the end-point bias of the HP filter, is based on augmenting the sample with forecasts from surveys or econometric models; see Kaiser, R. and Maravall, A. (1999); Mise, E., Kim, T.-H. and Newbold, P. (2005). However, the resulting estimates of the trend are dependant on the accuracy of the model forecasts about which consensus may not exist. Given the uncertainty surrounding such forecasts, there is a risk that end-point biases remain substantial.

In this section we propose a different solution to the end-point bias of the HP filter. Our solution is to assign a different smoothing parameter, α_1, to the end-weights $p_{n,j}$, $j = 1, \ldots, n$, in the expression from Corollary 2.1. In order to reduce the end-point bias, we need $\alpha_1 > \alpha$, since a large smoothing parameter reduces the weights; see (2.25) and the discussion from the end of Section 2. Note that without knowing the explicit analytical expression for the weights, it is impossible to use a different smoothing parameter for each weight; see Corollary 2.1 and equation (A.1) from the beginning of the proof of Theorem 2.1.

We illustrate our solution on the estimation of the turning points dated by NBER. Our data consists of seasonally adjusted quarterly U.S. real GDP from 1947:Q1 to 2013:Q2 taken from Philadelphia's Fed Real-Time Data Research Center (a thoroughly documented data set of 266 observations using a unique vintage of data from 2013:Q3; Croushore, D. and Stark, T. (2001)). Tables 2 and 3 show the NBER recession dates and the recession dates estimated by the HP filter using the whole sample of observations, $n = 266$, with $\alpha = 1600$ (as suggested by Hodrick, R. and Prescott, E. (1997)) for all the weights $p_{i,j}$, $i, j = 1, \ldots, n$ (second column); using the sample recursively with $n = 20, 21, \ldots, 266$ and $\alpha = 1600$ for all the weights (third column); and using the sample recursively with $n = 20, 21, \ldots, 266$ and $\alpha = 1600$ for all the weights, except $p_{n,j}$, $j = 1, \ldots, n$, for which the smoothing parameter is $\alpha_1 = 150,000$ (last column). The estimates of the troughs and peaks by the HP filter are based on a very standard and widely used rule; see Canova, F. (1999); Wecker, W. (1979); Zellner, A. and Hong, C. (1991). It defines a trough as a situation where two consecutive declines in the cyclical component of the GDP are followed by an increase, i.e., at time i, $\hat{c}_{i+1} > \hat{c}_i < \hat{c}_{i-1} < \hat{c}_{i-2}$. A peak is defined as a situation where two consecutive increases in the cyclical component of the GDP are followed by a decline, i.e., at time i, $\hat{c}_{i+1} < \hat{c}_i > \hat{c}_{i-1} > \hat{c}_{i-2}$.

Table 2: NBER peak dates and estimated peaks by HP filter

NBER	HP full sample	HP recursive	HP recursive corrected
1953:Q2	1953:Q1	1953:Q1	1953:Q1
1957:Q3	1957:Q1	1956:Q4	1955:Q3
1960:Q2	1960:Q1	1960:Q1	1960:Q1
1969:Q4	1969:Q1	1968:Q2	1968:Q2
1973:Q4	1973:Q2	1973:Q1	1973:Q2
1980:Q1	1978:Q4	1978:Q2	1978:Q4
1981:Q3	1981:Q1	1981:Q1	1981:Q1
1990:Q3	1990:Q1	1987:Q4	1989:Q2
2001:Q1	2000:Q2	1999:Q4	2000:Q2
2007:Q4	2007:Q4	2007:Q3	2007:Q3

As it can be noted from the second column of Tables 2 and 3, the HP filter based on the full sample is very reliable in identifying all the turning points dated by NBER. This is the same conclusion as in Canova, F. (1999).[3] However, Canova's results are based on the full sample of observations and are not affected by the end of sample bias of the filter, about which policy-makers are more concerned with, given

[3]Canova, F. (1999) compares the HP filter with many other detrending methods (model-based detrending, linear detrending, Beveridge-Nelson decomposition, unobservable components, a frequency domain procedure introduced by Sims, C. (1974) and its time dimension version, the band-pass filter, introduced by Baxter, M. and King, R. G. (1999), etc) and concludes that the HP filter and Baxter and King's filter are the best in mimicking NBER cycles regardless of the dating rule used. Moreover, Nilsson, R. and Gyomai, G. (2011) conclude that the HP filter outperforms the filter proposed by Christiano, L. and Fitzgerald, T. J. (2003) in estimating turning points.

Table 3: NBER trough dates and estimated troughs by HP filter

NBER	HP full sample	HP recursive	HP recursive corrected
1954:Q2	1954:Q2	1954:Q2	
1958:Q2	1958:Q2	1958:Q1	1958:Q1
1961:Q1	1961:Q1	1960:Q4	1960:Q4
1970:Q4	1970:Q4	1970:Q4	1970:Q4
1975:Q1	1975:Q1	1975:Q1	1975:Q1
1980:Q3	1980:Q3	1980:Q3	1980:Q3
1982:Q4	1982:Q4	1982:Q1	1982:Q4
1991:Q1	1991:Q1	1991:Q1	1991:Q1
2001:Q4	2001:Q4	2001:Q4	2001:Q4
2009:Q2	2009:Q2	2009:Q1	2009:Q2

the importance of the more recent observations in indicating the beginning (end) of a recession. Unfortunately, in the interesting case when the HP filter is applied recursively with the same smoothing parameter for all the weights, $\alpha = 1600$, the estimates of the turning points are very bad, as it can be seen from column three of Tables 2 and 3.[4] If instead we use a different smoothing parameter for the end-weights, $\alpha_1 = 150000$, the estimates are on average better, as indicated by the shaded boxes in column 3 of the tables. In fact, in four cases, the corrected recursive HP filter is able to estimate a peak closer to the NBER peak compared to the recursive HP filter. Only in one case it does worse than the recursive HP filter (for the NBER peak in 1957:Q3). Moreover, in two cases, the corrected recursive HP filter is able to estimate a trough closer to the NBER trough compared to the recursive HP filter.

Our solution of assigning a different smoothing parameter to the end-weights can be adapted to applications other than the estimation of turning points. For example, it could be used to reduce the end-point bias in the computation of the moments in GMM estimation and related statistics; see Christiano, L. and den Haan, W. (1996). Moreover, given the controversy about the best choice of the smoothing parameter (Backus, D.K. and Kehoe, P. J. (1992); Baxter, M. and King, R. G. (1999); Giorno, C.L., Richardson, P., Roseveare, D. and van den Noord, P. (1995); Ravn, M. O. and Uhlig, H. (2002)), a data dependent method similar to the GCV, but which takes into account the potential weak dependence in the filtered data (De Jong, R. M. and Sakarya, N. (2013); King, R. G. and Rebelo, S. T. (1993)) and also aims to reduce the end-point bias, can be derived based on our formulae for the weights.[5]

[4]The end-point bias is not a characteristic of the HP filter only, but also of the Baxter-King and Christiano-Fitzgerald filters; see Baxter, M. and King, R. G. (1999); Christiano, L. and Fitzgerald, T. J. (2003); Watson, M. W. (2007).

[5]Note that the GCV is underestimating the smoothing parameter if the filtered data is weakly dependant; see Christiano, L. and den Haan, W. (1996).

5 A methodology for computing the exact weights of the Whittaker-Henderson filters

The rationale that have led to Theorem 2.1 can be applied to derive the explicit formula for the weights for the more general filters (1.4) and (1.5). Note that the explicit solution to (1.4) is given by

$$\hat{\boldsymbol{\tau}} = (\boldsymbol{I}_n + \alpha \boldsymbol{F})^{-1} \boldsymbol{y}, \tag{5.1}$$

where

$$\boldsymbol{F} = \boldsymbol{Q}^r - \boldsymbol{Z}\boldsymbol{Z}' - \boldsymbol{P}_n \boldsymbol{Z}\boldsymbol{Z}'\boldsymbol{P}_n, \tag{5.2}$$

$\boldsymbol{P}_n$ and $\boldsymbol{Q}$ are defined in (2.3) and (2.2), and $\boldsymbol{Z}$ is a matrix of size $n \times (r-1)$ given by

$$\boldsymbol{Z}' := \left(\begin{array}{cc} \boldsymbol{Z}_1' & \mathbf{O}_{n-r,r-1}' \end{array} \right), \tag{5.3}$$

where $\boldsymbol{Z}_1'$ is a matrix of size $(r-1) \times r$ and of rank $r-1$. For example, for $p = 3$ we have

$$\boldsymbol{Z}' := \left(\left(\begin{array}{ccc} 3 & -3 & 1 \\ 2 & -1 & 0 \end{array} \right) \quad \mathbf{O}_{2,n-3} \right). \tag{5.4}$$

As in Theorem 2.1, the solution $(\boldsymbol{I}_n + \alpha \boldsymbol{F})^{-1}$ can be expressed in terms of α, n and the eigenvalues/eigenvectors of $\boldsymbol{Q}$ only, by applying the Sherman-Morrison-Woodbury (SMW) formula twice, first to account for $\boldsymbol{P}_n \boldsymbol{Z}\boldsymbol{Z}'\boldsymbol{P}_n$, second to account for $\boldsymbol{Z}\boldsymbol{Z}'$; see Abadir, K. M. and Magnus, J. R. (2005), p.107, where their matrix $\boldsymbol{D}$ is the identity matrix $\boldsymbol{I}_{r-1}$ in our case. The SMW formula reduces the Sherman-Morrison formula when the rank of $\boldsymbol{Z}\boldsymbol{Z}'$ is 1 (for $r = 2$). To illustrate the SMW in this setting, denote by $\boldsymbol{A} := \boldsymbol{I}_n + \alpha \boldsymbol{Q}^r$ and by $\boldsymbol{C} := \boldsymbol{A} - \alpha \boldsymbol{Z}\boldsymbol{Z}'$. Then,

$$\begin{aligned} (\boldsymbol{I}_n + \alpha \boldsymbol{F})^{-1} &= (\boldsymbol{C} - \alpha \boldsymbol{P}_n \boldsymbol{Z}\boldsymbol{Z}'\boldsymbol{P}_n)^{-1} \qquad (5.5) \\ &= \boldsymbol{C}^{-1} + \alpha \boldsymbol{C}^{-1}\boldsymbol{P}_n \boldsymbol{Z} \left(\boldsymbol{I}_{r-1} - \alpha \boldsymbol{Z}'\boldsymbol{P}_n \boldsymbol{C}^{-1}\boldsymbol{P}_n \boldsymbol{Z} \right)^{-1} \boldsymbol{Z}'\boldsymbol{P}_n \boldsymbol{C}^{-1}. \end{aligned}$$

This is the SMW applied once. We now apply the SMW formula for $\boldsymbol{C}^{-1}$,

$$\boldsymbol{C}^{-1} = \boldsymbol{A}^{-1} + \alpha \boldsymbol{A}^{-1}\boldsymbol{Z} \left(\boldsymbol{I}_{r-1} - \alpha \boldsymbol{Z}'\boldsymbol{A}^{-1}\boldsymbol{Z} \right)^{-1} \boldsymbol{Z}'\boldsymbol{A}^{-1}. \tag{5.6}$$

Note that the matrices $\left(\boldsymbol{I}_{r-1} - \alpha \boldsymbol{Z}'\boldsymbol{P}_n \boldsymbol{C}^{-1}\boldsymbol{P}_n \boldsymbol{Z}\right)^{-1}$ and $\left(\boldsymbol{I}_{r-1} - \alpha \boldsymbol{Z}'\boldsymbol{A}^{-1}\boldsymbol{Z}\right)^{-1}$ are of size $(r-1) \times (r-1)$. To obtain $\left(\boldsymbol{I}_{r-1} - \alpha \boldsymbol{Z}'\boldsymbol{A}^{-1}\boldsymbol{Z}\right)^{-1}$, we can apply a few times either the SMW formula or the formula for partitioned matrix (Abadir, K. M. and Magnus, J. R. (2005), p.106) by exploiting the zeros in $\boldsymbol{Z}$. Simplifications along the lines of the proof of Theorem 2.1, as for example in (A.3), (A.5), (A.6), (A.7), occur.

To invert $\boldsymbol{A}$, it is enough to know the eigenvalues of $\boldsymbol{Q}$ (given in (2.4)) in order to derive the eigenvalues of $\boldsymbol{Q}^r$ and functions of them. More exactly, since $\boldsymbol{Q} = \boldsymbol{T}\boldsymbol{\Gamma}\boldsymbol{T}$ (because eigenvalues in $\boldsymbol{\Gamma} = \text{diag}(\gamma_1, \cdots, \gamma_n)$ are distinct; see (2.4)), we have that $\boldsymbol{Q}^r = \boldsymbol{T}\boldsymbol{\Gamma}^r\boldsymbol{T}$ (Abadir, K. M. and Magnus, J. R. (2005), p.245). Hence $\boldsymbol{A}^{-1} := \boldsymbol{T}\boldsymbol{\Lambda}^{-1}\boldsymbol{T}$, where

$$\boldsymbol{\Lambda}^{-1} := \text{diag}\left(\lambda_1^{-1}, \ldots, \lambda_j^{-1}, \ldots, \lambda_n^{-1}\right), \; \lambda_j = 1 + \alpha \gamma_j^r, \tag{5.7}$$

with γ_j given in (2.4).

To obtain the exact solution for the filter in (1.5), we have to proceed in the similar manner described above, and apply SMW formula twice, times the number of penalty terms, in order to reduce the minimization problem (1.5) to the inversion of matrices of size $(r-1)\times(r-1)$, instead of matrices of size $n\times n$, where $r << n$.

6 Conclusion

In this paper we obtain the exact analytical expression for the weights in the HP filter, a result that has not been previously derived in the literature. We then use the expression for the weights to build a fast algorithm that can be implemented in software. Our algorithm is up to fifty times faster for sample sizes typical in economics. Moreover, our expression for the weights gives insights about the properties of the HP filter and we use it to propose a solution to the end-point bias of the filter. We illustrate the performance of our solution on the estimation of the peaks and troughs dated by the NBER. We show that our solution for the end-point bias gives better estimates of the recession dates compared to the usual HP filter. Finally, we show that our derivations for the weights provide a methodology for finding the exact weights of the more general Whittaker-Henderson filters of which the HP filter is a particular case. Other than these applications of our formulae for the weights, our results can also be used to derive analytically the moments needed in the estimation of DSGE models; to propose a solution for reducing spurious correlations/cycles and the problems these induce for inference, and to propose a data-dependent method for the choice of the smoothing parameter that is valid with weakly dependant detrended data.

A Appendix section

A.1 Proof of Theorem 2.1

Denote $\boldsymbol{C} := \boldsymbol{Q}\boldsymbol{Q}$ and $\boldsymbol{A} := \boldsymbol{I}_n + \alpha\boldsymbol{C}$. We apply Sherman-Morrison formula twice:

$$\begin{aligned}
(\boldsymbol{I}_n+\alpha\boldsymbol{F})^{-1} &= (\boldsymbol{A}-\alpha\boldsymbol{g}\boldsymbol{g}'-\alpha\boldsymbol{P}_n\boldsymbol{g}\boldsymbol{g}'\boldsymbol{P}_n)^{-1} \\
&= (\boldsymbol{A}-\alpha\boldsymbol{g}\boldsymbol{g}')^{-1}+\alpha\frac{(\boldsymbol{A}-\alpha\boldsymbol{g}\boldsymbol{g}')^{-1}\boldsymbol{P}_n\boldsymbol{g}\boldsymbol{g}'\boldsymbol{P}_n(\boldsymbol{A}-\alpha\boldsymbol{g}\boldsymbol{g}')^{-1}}{1-\alpha\boldsymbol{g}'\boldsymbol{P}_n(\boldsymbol{A}-\alpha\boldsymbol{g}\boldsymbol{g}')^{-1}\boldsymbol{P}_n\boldsymbol{g}} \\
&= \boldsymbol{A}^{-1}+\alpha\frac{\boldsymbol{A}^{-1}\boldsymbol{g}\boldsymbol{g}'\boldsymbol{A}^{-1}}{1-\alpha\boldsymbol{g}'\boldsymbol{A}^{-1}\boldsymbol{g}} \\
&\quad +\alpha\frac{\left(\boldsymbol{A}^{-1}+\alpha\frac{\boldsymbol{A}^{-1}\boldsymbol{g}\boldsymbol{g}'\boldsymbol{A}^{-1}}{1-\alpha\boldsymbol{g}'\boldsymbol{A}^{-1}\boldsymbol{g}}\right)\boldsymbol{P}_n\boldsymbol{g}\boldsymbol{g}'\boldsymbol{P}_n\left(\boldsymbol{A}^{-1}+\alpha\frac{\boldsymbol{A}^{-1}\boldsymbol{g}\boldsymbol{g}'\boldsymbol{A}^{-1}}{1-\alpha\boldsymbol{g}'\boldsymbol{A}^{-1}\boldsymbol{g}}\right)}{1-\alpha\boldsymbol{g}'\boldsymbol{P}_n\left(\boldsymbol{A}^{-1}+\alpha\frac{\boldsymbol{A}^{-1}\boldsymbol{g}\boldsymbol{g}'\boldsymbol{A}^{-1}}{1-\alpha\boldsymbol{g}'\boldsymbol{A}^{-1}\boldsymbol{g}}\right)\boldsymbol{P}_n\boldsymbol{g}}.
\end{aligned} \tag{A.1}$$

We have $\boldsymbol{A} := \boldsymbol{I}_n + \alpha\boldsymbol{C} = \boldsymbol{I}_n + \alpha\boldsymbol{Q}\boldsymbol{Q}$. Since the tridiagonal matrix $\boldsymbol{Q}$ has distinct eigenvalues given in (2.4), $\boldsymbol{Q}$ can be written as $\boldsymbol{Q} := \boldsymbol{T}\boldsymbol{\Gamma}\boldsymbol{T}^{-1}$, where $\boldsymbol{\Gamma} :=$

diag $(\gamma_1, \ldots, \gamma_j, \ldots, \gamma_n)$. Also, $\boldsymbol{QQ} = \boldsymbol{T}\text{diag}\left(\gamma_1^2, \ldots, \gamma_j^2, \ldots, \gamma_n^2\right)\boldsymbol{T}^{-1}$. The typical element of the matrix $\boldsymbol{T}$ is given in (2.5).

The constant $\left(\frac{2}{n+1}\right)^{1/2} = \left(\sum_{i=1}^{n} \sin^2\left(\frac{\pi i}{n+1}\right)\right)^{-1/2}$ guarantees that the eigenvectors of $\boldsymbol{Q}$ are orthonormal, in which case $\boldsymbol{Q} := \boldsymbol{T\Gamma T}' = \boldsymbol{T\Gamma T}$, by the symmetry of $\boldsymbol{T}$.

The matrix $\boldsymbol{A}$ has eigenvalues given by (2.7) and the same eigenvectors as $\boldsymbol{Q}$ given by (2.5). Hence $\boldsymbol{A} := \boldsymbol{T\Lambda T}$, where $\boldsymbol{\Lambda}$ is as in (2.6). Hence, $\boldsymbol{A}^{-1} := \boldsymbol{T\Lambda}^{-1}\boldsymbol{T}$.

Denote $\boldsymbol{u} := (u_1, u_2, \cdots, u_n)'$, where

$$u_j := \sum_{s=1}^{n} \frac{2x_{1,s} - x_{2,s}}{\lambda_s} x_{s,j},\ j = 1, \cdots, n, \tag{A.2}$$

and note that

$$\boldsymbol{gg}'\boldsymbol{A}^{-1} := \begin{pmatrix} 2\boldsymbol{u} \\ -\boldsymbol{u} \\ \boldsymbol{O}_{n-2,n} \end{pmatrix},\quad \boldsymbol{P}_n\boldsymbol{gg}'\boldsymbol{P}_n\boldsymbol{A}^{-1} := \begin{pmatrix} \boldsymbol{O}_{n-2,n} \\ -\boldsymbol{P}_n\boldsymbol{u} \\ 2\boldsymbol{P}_n\boldsymbol{u} \end{pmatrix}$$

which follows by noting that $x_{i,j} = x_{j,i}$ and $\sin\left(\frac{(n-1)j\pi}{n+1}\right)\sin\left(\frac{j\pi}{n+1}\right) = \sin\left(\frac{2j\pi}{n+1}\right)\sin\left(\frac{nj\pi}{n+1}\right)$. The matrices $\boldsymbol{gg}'\boldsymbol{A}^{-1}$, $\boldsymbol{P}_n\boldsymbol{gg}'\boldsymbol{P}_n\boldsymbol{A}^{-1}$ and $\boldsymbol{P}_n\boldsymbol{gg}'\boldsymbol{P}_n\boldsymbol{A}^{-1}\boldsymbol{gg}'\boldsymbol{A}^{-1}$ have only one nonzero eigenvalue given by

$$\boldsymbol{g}'\boldsymbol{A}^{-1}\boldsymbol{g} = \boldsymbol{g}'\boldsymbol{P}_n\boldsymbol{A}^{-1}\boldsymbol{P}_n\boldsymbol{g} = 2u_1 - u_2 = \sum_{s=1}^{n} \frac{(2x_{1,s} - x_{2,s})^2}{\lambda_s}. \tag{A.3}$$

$$\begin{aligned} \boldsymbol{g}'\boldsymbol{P}_n\boldsymbol{A}^{-1}\boldsymbol{gg}'\boldsymbol{A}^{-1}\boldsymbol{P}_n\boldsymbol{g} &= (2u_n - u_{n-1})^2 \\ &= \left(\sum_{s=1}^{n} \frac{(2x_{1,s} - x_{2,s})^2}{\lambda_s}(-1)^{s-1}\right)^2, \end{aligned} \tag{A.4}$$

by noting that $x_{s,n} = x_{s,1}(-1)^{s-1}$ and $x_{s,n-1} = x_{s,2}(-1)^{s-1}$. Hence

$$\begin{aligned} (\boldsymbol{I}_n + \alpha\boldsymbol{F})^{-1} &= \boldsymbol{A}^{-1} + \frac{\alpha\boldsymbol{A}^{-1}\boldsymbol{gg}'\boldsymbol{A}^{-1}}{1 - \alpha(2u_1 - u_2)} \\ &+ \frac{\alpha(1 - \alpha(2u_1 - u_2))\boldsymbol{A}^{-1}\boldsymbol{P}_n\boldsymbol{gg}'\boldsymbol{P}_n\boldsymbol{A}^{-1}}{(1 - \alpha(2u_1 - u_2))^2 - \alpha^2(2u_n - u_{n-1})^2} \\ &+ \frac{\alpha^2\boldsymbol{A}^{-1}(\boldsymbol{P}_n\boldsymbol{gg}'\boldsymbol{P}_n\boldsymbol{A}^{-1}\boldsymbol{gg}' + \boldsymbol{gg}'\boldsymbol{A}^{-1}\boldsymbol{P}_n\boldsymbol{gg}'\boldsymbol{P}_n)\boldsymbol{A}^{-1}}{(1 - \alpha(2u_1 - u_2))^2 - \alpha^2(2u_n - u_{n-1})^2} \\ &+ \frac{\alpha^2\boldsymbol{A}^{-1}\boldsymbol{gg}'\boldsymbol{A}^{-1}\boldsymbol{P}_n\boldsymbol{gg}'\boldsymbol{P}_n\boldsymbol{A}^{-1}\boldsymbol{gg}'\boldsymbol{A}^{-1}}{(1 - \alpha(2u_1 - u_2))^2 - \alpha^2(2u_n - u_{n-1})^2}\frac{\alpha}{1 - \alpha(2u_1 - u_2)}. \end{aligned}$$

Notice that

$$\boldsymbol{gg}'\boldsymbol{A}^{-1}\boldsymbol{P}_n\boldsymbol{gg}'\boldsymbol{P}_n\boldsymbol{A}^{-1}\boldsymbol{gg}' = (2u_n - u_{n-1})^2\boldsymbol{gg}', \tag{A.5}$$

$$\boldsymbol{g}\boldsymbol{g}'\boldsymbol{A}^{-1}\boldsymbol{P}_n\boldsymbol{g}\boldsymbol{g}'\boldsymbol{P}_n = -(2u_n - u_{n-1})\boldsymbol{g}\boldsymbol{g}'\boldsymbol{P}_n, \tag{A.6}$$

$$\boldsymbol{P}_n\boldsymbol{g}\boldsymbol{g}'\boldsymbol{P}_n\boldsymbol{A}^{-1}\boldsymbol{g}\boldsymbol{g}' = -(2u_n - u_{n-1})\boldsymbol{P}_n\boldsymbol{g}\boldsymbol{g}'. \tag{A.7}$$

Hence

$$\begin{aligned}
(\boldsymbol{I}_n + \alpha\boldsymbol{F})^{-1} &= \boldsymbol{A}^{-1} + \frac{\alpha\boldsymbol{A}^{-1}\boldsymbol{g}\boldsymbol{g}'\boldsymbol{A}^{-1}}{1-\alpha(2u_1-u_2)} + \frac{\alpha(1-\alpha(2u_1-u_2))\boldsymbol{A}^{-1}\boldsymbol{P}_n\boldsymbol{g}\boldsymbol{g}'\boldsymbol{P}_n\boldsymbol{A}^{-1}}{(1-\alpha(2u_1-u_2))^2 - \alpha^2(2u_n-u_{n-1})^2} \\
&\quad - \frac{\alpha^2(2u_n-u_{n-1})\boldsymbol{A}^{-1}(\boldsymbol{P}_n\boldsymbol{g}\boldsymbol{g}' + \boldsymbol{g}\boldsymbol{g}'\boldsymbol{P}_n)\boldsymbol{A}^{-1}}{(1-\alpha(2u_1-u_2))^2 - \alpha^2(2u_n-u_{n-1})^2} \\
&\quad + \frac{\alpha}{1-\alpha(2u_1-u_2)}\frac{\alpha^2(2u_n-u_{n-1})^2\boldsymbol{A}^{-1}\boldsymbol{g}\boldsymbol{g}'\boldsymbol{A}^{-1}}{(1-\alpha(2u_1-u_2))^2 - \alpha^2(2u_n-u_{n-1})^2} \\
&= \boldsymbol{A}^{-1} + \frac{\alpha(1-\alpha(2u_1-u_2))\boldsymbol{A}^{-1}(\boldsymbol{g}\boldsymbol{g}' + \boldsymbol{P}_n\boldsymbol{g}\boldsymbol{g}'\boldsymbol{P}_n)\boldsymbol{A}^{-1}}{(1-\alpha(2u_1-u_2))^2 - \alpha^2(2u_n-u_{n-1})^2} \\
&\quad - \frac{\alpha^2(2u_n-u_{n-1})\boldsymbol{A}^{-1}(\boldsymbol{P}_n\boldsymbol{g}\boldsymbol{g}' + \boldsymbol{g}\boldsymbol{g}'\boldsymbol{P}_n)\boldsymbol{A}^{-1}}{(1-\alpha(2u_1-u_2))^2 - \alpha^2(2u_n-u_{n-1})^2} \\
&=: \boldsymbol{A}^{-1} + \boldsymbol{T}\boldsymbol{\Lambda}^{-1}\boldsymbol{H}\boldsymbol{\Lambda}^{-1}\boldsymbol{T},
\end{aligned}$$

where

$$\boldsymbol{H} := \boldsymbol{T}\left[(\boldsymbol{g}\boldsymbol{g}' + \boldsymbol{P}_n\boldsymbol{g}\boldsymbol{g}'\boldsymbol{P}_n)w_1 + (\boldsymbol{P}_n\boldsymbol{g}\boldsymbol{g}' + \boldsymbol{g}\boldsymbol{g}'\boldsymbol{P}_n)w_2\right]\boldsymbol{T},$$

$$\begin{aligned}
w_1 &:= \frac{\alpha(1-\alpha(2u_1-u_2))}{(1-\alpha(2u_1-u_2))^2 - \alpha^2(2u_n-u_{n-1})^2}, \\
w_2 &:= -\frac{\alpha^2(2u_n-u_{n-1})}{(1-\alpha(2u_1-u_2))^2 - \alpha^2(2u_n-u_{n-1})^2},
\end{aligned}$$

with u_1, u_2, u_{n-1}, u_n given in (A.2).

It is easy to compute $\boldsymbol{T}\boldsymbol{g}\boldsymbol{g}'\boldsymbol{T}, \boldsymbol{T}\boldsymbol{P}_n\boldsymbol{g}\boldsymbol{g}'\boldsymbol{P}_n\boldsymbol{T}, \boldsymbol{T}\boldsymbol{g}\boldsymbol{g}'\boldsymbol{P}_n\boldsymbol{T}, \boldsymbol{T}\boldsymbol{P}_n\boldsymbol{g}\boldsymbol{g}'\boldsymbol{T}$, typical elements of which for row i and column j are:

$$\begin{aligned}
&(2x_{i,1} - x_{i,2})(2x_{1,j} - x_{2,j}), \\
&(2x_{i,n} - x_{i,n-1})(2x_{n,j} - x_{n-1,j}), \\
&(2x_{i,1} - x_{i,2})(2x_{n,j} - x_{n-1,j}), \\
&(2x_{i,n} - x_{i,n-1})(2x_{1,j} - x_{2,j}),
\end{aligned}$$

respectively, for $i, j = 1, \ldots, n$. Hence the typical element of $\boldsymbol{K} := \boldsymbol{\Lambda}^{-1}\boldsymbol{H}\boldsymbol{\Lambda}^{-1}$ for row i and column j is

$$\begin{aligned}
h_{i,j} &:= \frac{(2x_{i,1} - x_{i,2})(2x_{1,j} - x_{2,j})w_1 + (2x_{i,n} - x_{i,n-1})(2x_{n,j} - x_{n-1,j})w_1}{\lambda_i\lambda_j} \\
&\quad + \frac{(2x_{i,1} - x_{i,2})(2x_{n,j} - x_{n-1,j})w_2 + (2x_{i,n} - x_{i,n-1})(2x_{1,j} - x_{2,j})w_2}{\lambda_i\lambda_j}.
\end{aligned}$$

For $i = j$

$$h_{i,i} := \frac{(2x_{i,1} - x_{i,2})^2 w_1 + (2x_{i,n} - x_{i,n-1})^2 w_1 + 2(2x_{i,1} - x_{i,2})(2x_{i,n} - x_{i,n-1}) w_2}{\lambda_i^2}.$$

Note that $x_{i,n} = x_{i,1}(-1)^{i-1}$, $x_{i,n-1} = x_{i,2}(-1)^{i-1}$, $x_{n,j} = x_{1,j}(-1)^{j-1}$ and $x_{n-1,j} = x_{2,j}(-1)^{j-1}$. Hence

$$\begin{aligned}
h_{i,j} &:= \frac{(2x_{i,1} - x_{i,2})(2x_{1,j} - x_{2,j}) w_1 + (2x_{i,1} - x_{i,2})(2x_{1,j} - x_{2,j})(-1)^{i-1}(-1)^{j-1} w_1}{\lambda_i \lambda_j} \\
&+ \frac{(2x_{i,1} - x_{i,2})(2x_{1,j} - x_{2,j})(-1)^{j-1} w_2 + (2x_{i,1} - x_{i,2})(2x_{1,j} - x_{2,j})(-1)^{i-1} w_2}{\lambda_i \lambda_j} \\
&= \frac{(2x_{i,1} - x_{i,2})(2x_{1,j} - x_{2,j})}{\lambda_i \lambda_j}\left[w_1\left[1 + (-1)^{i-1}(-1)^{j-1}\right] + w_2\left[(-1)^{j-1} + (-1)^{i-1}\right]\right] \\
&= \frac{(2x_{i,1} - x_{i,2})(2x_{1,j} - x_{2,j})}{\lambda_i \lambda_j}\left[1 + (-1)^{i+j-2}\right]\left[w_1 + (-1)^{1-j} w_2\right] \\
&= \begin{cases} 0 & \text{, if } i+j \text{ odd,} \\ 2(2x_{i,1} - x_{i,2})(2x_{1,j} - x_{2,j})\lambda_i^{-1}\lambda_j^{-1}[w_1 - w_2], & \text{if } i+j \text{ even and } j \text{ even,} \\ 2(2x_{i,1} - x_{i,2})(2x_{1,j} - x_{2,j})\lambda_i^{-1}\lambda_j^{-1}[w_1 + w_2], & \text{if } i+j \text{ even and } j \text{ odd.} \end{cases}
\end{aligned}$$

For $i = j$

$$\begin{aligned}
h_{i,i} &:= \frac{(2x_{i,1} - x_{i,2})^2}{\lambda_i^2}\left[1 + (-1)^{2(i-1)}\right]\left[w_1 + (-1)^{1-i} w_2\right] \\
&= 2\frac{(2x_{i,1} - x_{i,2})^2}{\lambda_i^2}\left[w_1 + (-1)^{1-i} w_2\right] \\
&= \begin{cases} 2(2x_{i,1} - x_{i,2})^2 \lambda_i^{-2}[w_1 + w_2], & \text{if } i \text{ odd,} \\ 2(2x_{i,1} - x_{i,2})^2 \lambda_i^{-2}[w_1 - w_2], & \text{if } i \text{ even.} \end{cases}
\end{aligned}$$

We also need

$$\begin{aligned}
w_1 - w_2 &= \frac{\alpha(1 - \alpha(2u_1 - u_2)) + \alpha^2(2u_n - u_{n-1})}{(1 - \alpha(2u_1 - u_2))^2 - \alpha^2(2u_n - u_{n-1})^2} \\
&= \frac{\alpha(1 - \alpha(2u_1 - u_2)) + \alpha^2(2u_n - u_{n-1})}{(1 - \alpha(2u_1 - u_2) + \alpha(2u_n - u_{n-1}))(1 - \alpha(2u_1 - u_2) - \alpha(2u_n - u_{n-1}))} \\
&= \frac{\alpha}{1 - \alpha\sum_{s=1}^{n}(2x_{1,s} - x_{2,s})^2 \lambda_s^{-1}\left[1 - (-1)^{s-1}\right]}, \\
w_1 + w_2 &= \frac{\alpha}{1 - \alpha(2u_1 - u_2) + \alpha(2u_n - u_{n-1})} \\
&= \frac{\alpha}{1 - \alpha\sum_{s=1}^{n}(2x_{1,s} - x_{2,s})^2 \lambda_s^{-1}\left[1 + (-1)^{s-1}\right]}.
\end{aligned}$$

Hence,

$$\boldsymbol{K} := \frac{2\alpha \boldsymbol{K}_1}{1-\alpha\sum_{s=1}^{n}(2x_{1,s}-x_{2,s})^2\lambda_s^{-1}\left[1+(-1)^{s-1}\right]} + \frac{2\alpha \boldsymbol{K}_2}{1-\alpha\sum_{s=1}^{n}(2x_{1,s}-x_{2,s})^2\lambda_s^{-1}\left[1-(-1)^{s-1}\right]},$$

with $\boldsymbol{K}_1$ and $\boldsymbol{K}_2$ $n \times n$ matrices with typical element given by (2.10), as described on p.5. Finally,

$$(\boldsymbol{I}_n+\alpha\boldsymbol{F})^{-1} = \boldsymbol{T}\boldsymbol{\Lambda}^{-1}\boldsymbol{T} + \frac{2\alpha}{1-2\alpha\sum\limits_{j=1,\mathrm{odd}}^{n}(2x_{1,j}-x_{2,j})^2\lambda_j^{-1}}\boldsymbol{T}\boldsymbol{K}_1\boldsymbol{T} + \frac{2\alpha}{1-2\alpha\sum\limits_{j=1,\mathrm{even}}^{n}(2x_{1,j}-x_{2,j})^2\lambda_j^{-1}}\boldsymbol{T}\boldsymbol{K}_2\boldsymbol{T}.$$

Q.E.D.

A.2 Proof of Corollary 2.2

The proof follows by using the fact that the eigenvalues defined in (2.7) can also be written as

$$\lambda_s = 1+16\alpha\left(\sin\left(\frac{s\pi}{2(n+1)}\right)\right)^4, \quad s=1,\dots,n.$$

Using this, we have

$$\sum_{s=1}^{n}\frac{(2x_{1,s}-x_{2,s})^2}{\lambda_s} = \frac{2}{n+1}\sum_{s=1}^{n}\frac{16\left(\sin\left(\frac{s\pi}{2(n+1)}\right)\right)^4\left(\sin(\frac{s\pi}{n+1})\right)^2}{1+16\alpha\left(\sin\left(\frac{s\pi}{2(n+1)}\right)\right)^4}.$$

Let $r=s/(n+1)$, (2.21) follows by Lemma 3 of De Jong, R. M. and Sakarya, N. (2013). Also note that

$$\sum_{s=1,\ \mathrm{even}}^{n}\frac{(2x_{1,s}-x_{2,s})^2}{\lambda_s} = \frac{2}{n+1}\sum_{s=1}^{[n/2]}\frac{16\left(\sin\left(\frac{2s\pi}{2(n+1)}\right)\right)^4\left(\sin(\frac{2s\pi}{n+1})\right)^2}{1+16\alpha\left(\sin\left(\frac{2s\pi}{2(n+1)}\right)\right)^4}.$$

Let $r=s/(n+1)$. Then the result in (2.20) for s even, follows by Lemma 3 of De Jong, R. M. and Sakarya, N. (2013). For s odd, we have

$$\sum_{s=1,\ \mathrm{odd}}^{n}\frac{(2x_{1,s}-x_{2,s})^2}{\lambda_s} = \frac{2}{n+1}\sum_{s=0}^{\left[\frac{n-1}{2}\right]}\frac{16\left(\sin\left(\frac{(2s+1)\pi}{2(n+1)}\right)\right)^4\left(\sin(\frac{(2s+1)\pi}{n+1})\right)^2}{1+16\alpha\left(\sin\left(\frac{(2s+1)\pi}{2(n+1)}\right)\right)^4},$$

where by Lemma 3 of De Jong, R. M. and Sakarya, N. (2013) and with $r=(s+1/2)/(n+1)$, (2.20) follows. The same type of arguments can be used to show the result in (2.22). *Q.E.D*

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