Equilibria Extremal Optimization

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Solution concepts

Problem description:

- Game → players, strategies, payoffs;
- Multiobjective optimization → optimize multiple, conflicting objectives;

Solution concepts:

- Nash equilibrium (NE): no player can improve its payoff by unilateral deviation
- Berge equilibrium (BE): players maximize each others payoffs
- Pareto optimal solution/equilibrium: no objective can be increased without decreasing another objective

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Nash ascendancy relation

Compare two strategy profiles:

- compute $k(s,q) = card\{i \in N | u_i(q_i, s_{-i}) > u_i(s), q_i \neq s_i\}$
- s Nash ascends q if $k(s,q) < k(q,s)$
- s non-dominated with respect to the Nash ascendancy relation: $\nexists q \in S$ such that q Nash ascends s.
- Nash non-dominated = NE.

Probabilistic Nash ascendancy relation

Reduce the number of payoff function evaluations:

- only a percent p of players are tested:
- $I_p = \{i_1, i_2, ..., i_{n_p}\} \subset N$
- then $k_p(s, q, I_p) = card\{i \in I_p | u_i(s) < u_i(q_i, s_{-i}), s_i \neq q_i\}$.
- $s \in S$ $p-$Nash ascends $q \in S$ with respect to I_p if $k_p(s, q, I_p) < k_p(q, s, I_p)$.
- $s \in S$ p-non-dominated $\nexists q \in S$, $\nexists I_p \subset N$ such that q p-ascends s with respect to I_p.
- p-non-dominated = Nash equilibria

Berge generative relation

Definition (Berge-Zhukovskii)

A strategy profile $s^* \in S$ is a Berge-Zhukovskii equilibrium if the inequality

$$u_i(s^*) \geq u_i(s_i^*, s_{-i})$$

holds for each player $i = 1, ..., n$, and all $s_{-i} \in S_{-i}$.

Quality measures:

$$b(s, q) = card\{i \in N, u_i(s) < u_i(s_i, q_{-i}), s_{-i} \neq q_{-i}\},$$

$$b_\epsilon(s, q) = card\{i \in N, u_i(s) < u_i(s_i, q_{-i}) + \epsilon, s_{-i} \neq q_{-i}\},$$

The same mechanism $\rightarrow$ generative relation

Evolutionary detection - Nash equilibria

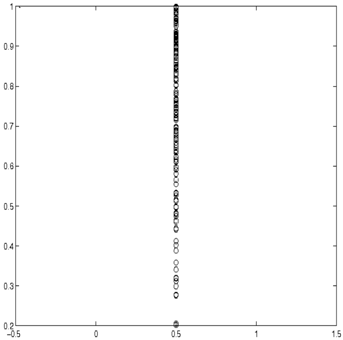
Game:

- Payoffs:

$$u_1(y_1, y_2) = y_1$$
$$u_2(y_1, y_2) = (0.5 - y_1)y_2$$

- $y_1 \in [0, 0.5]$, $y_2 \in [0, 1]$.
- NE: $(0.5, \lambda)$, $\lambda \in [0, 1]$.

NSGA-II results

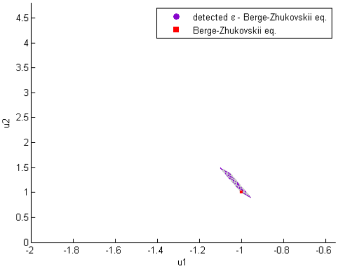


Evolutionary detection - Berge Zhukovskii equilibria

Example

$$u_1(s_1, s_2) = -s_1^2 - s_1 + s_2,$$
$$u_2(s_1, s_2) = 2s_1^2 + 3s_1 - s_2^2 - 3s_2,$$
$$s_i \in [-2, 1], i = 1, 2.$$

Berge-Zhukovskii equilibrium: $(1, 1)$;
Payoffs $(-1, 1)$.
$\rightarrow \epsilon \in \{0, 0.1, 0.2, 0.5, 0.9\}$.



Cournot oligopoly

- q_i, $i = 1, ..., N$ - quantities/ N companies
- market clearing price $P(Q) = a - Q$, where Q is the aggregate quantity on the market.
- total cost for the company i of producing quantity q_i: $C(q_i) = cq_i$.
- payoff for the company i is its profit:

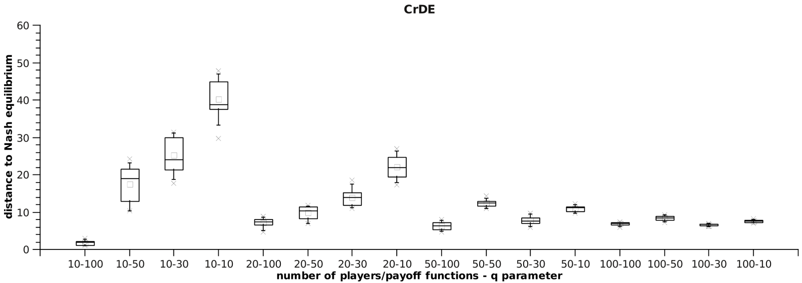
$$u_i(q_1, q_2, ..., q_N) = q_i P(Q) - C(q_i)$$

- if $Q = \sum_{i=1}^{N} q_i$, the Cournot oligopoly has one Nash equilibrium

$$q_i = \frac{a - c}{N + 1}, \forall i \in \{1, ..., N\}.$$

Multi-player games - probabilistic Nash ascendancy

Differential evolution



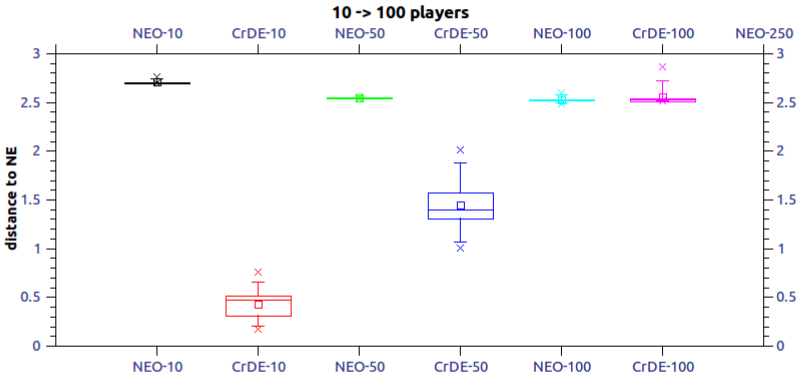
Nash Extremal Optimization

Algorithm 1 Nash Extremal Optimization procedure

```
Initialize configuration s = (s_1, ..., s_n) at will; set s_best := s;
repeat
    For the 'current' configuration s evaluate u_i for each player i;
    find j satisfying u_j ≤ u_i for all i, i ≠ j, i.e., j has the "worst payoff";
    change s_j randomly;
    if (s Nash ascends s_best) then
        set s_best := s;
    end if
until a termination condition is fulfilled;
Return s_best.
```

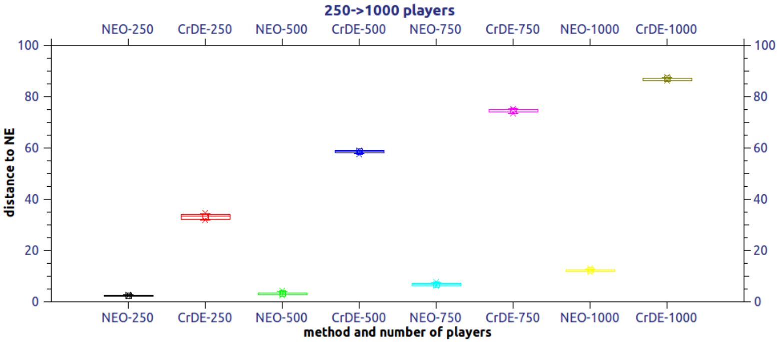
Multi-player games

Nash Extremal Optimization - Cournot

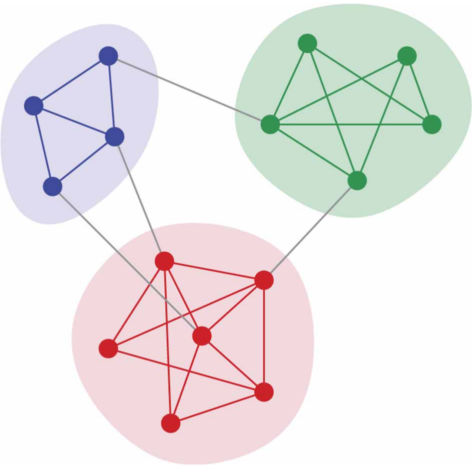


Multi-player games

Nash Extremal Optimization - Cournot

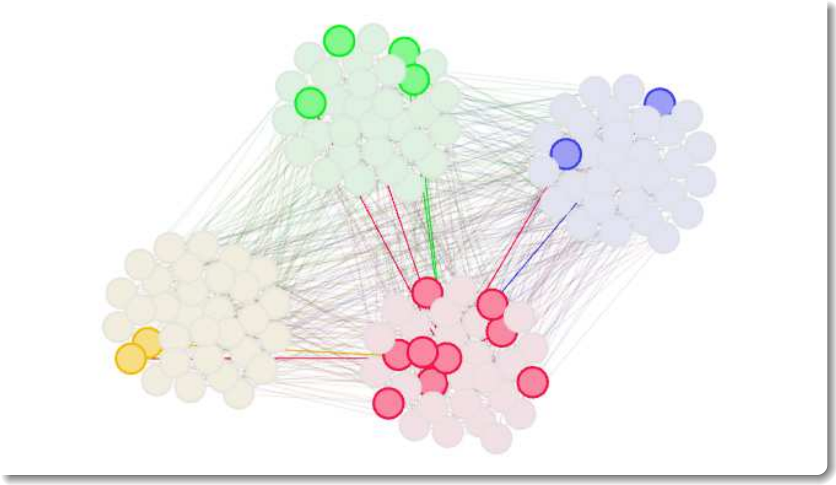


Community structure detection in social networks



M. E. J. Newman, Communities, modules and large-scale structure in networks, Nature Physics 8, 2531 (2012) doi:10.1038/nphys2162

Community structure - definitions



The game

Game $\Gamma = (N, S, U)$:

- $N = \{1, ..., n\}$, the set of players: network nodes;
- $S = S_1 \times S_2 \times \ldots \times S_n$, the set of strategy profiles of the game; $s \in S$, $s = (c_1, \ldots, c_n)$
- $U = \{u_i\}_{i=\overline{1,n}}$, the payoff functions; $u_i : S \to \mathbb{R}$ represents the payoff of player i, $i = \overline{1, n}$.

$$u_i(s) = f(C_i) - f(C_i \setminus \{i\}),$$

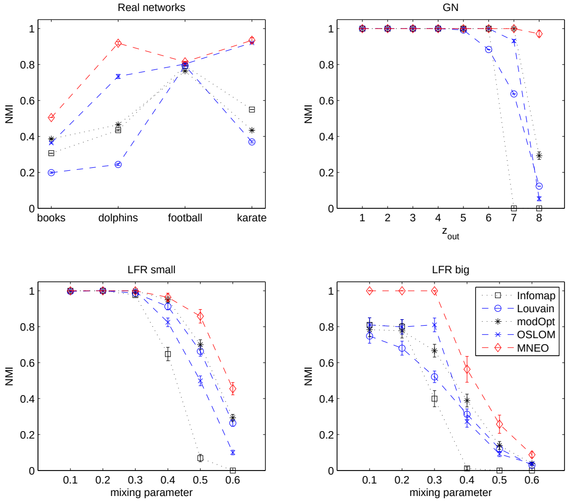
$$f(C) = \frac{k_{in}(C)}{(k_{in}(C) + k_{out}(C))^{\alpha}},$$

- $k_{in}(C)$ - double of the number of internal links in the community;
- $k_{out}(C)$ - number of external links of the community;
- α - a parameter controlling the size of the community.

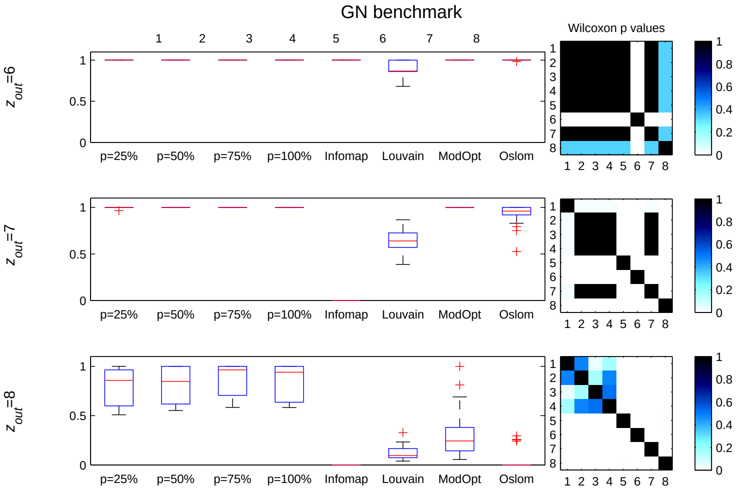
Mixed Network Extremal optimization

Randomly initialize all individuals in P and A;
Evaluate all individuals in P and A;
for $NrGen = 0$ to $MaxGen$ **do**
  Set $m_{ngen} = \max\left\{1, \left[\frac{1}{10} \cdot N \cdot (N-2)^{-\frac{NrGen}{MaxGen}}\right]\right\}$
  Run a NEO iteration for each pair (P_i, A_i);
  if Λ iterations were performed on the original network **then**
    Mix network with probability ρ;
    Randomly re-initialize population A;
    Evaluate all individuals in P and A;
  end if
  if Network is mixed and λ iterations were performed **then**
    Restore network to original structure;
    Randomly re-initialize population A;
    Evaluate all individuals in P and A;
  end if
end for
Output: individual from A having the best modularity;

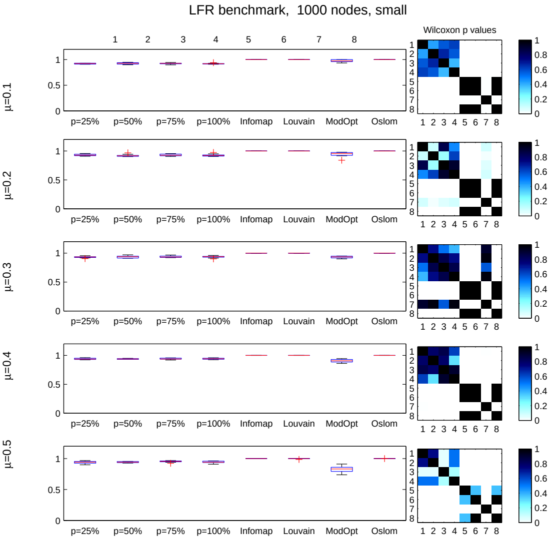
Results - MNEO



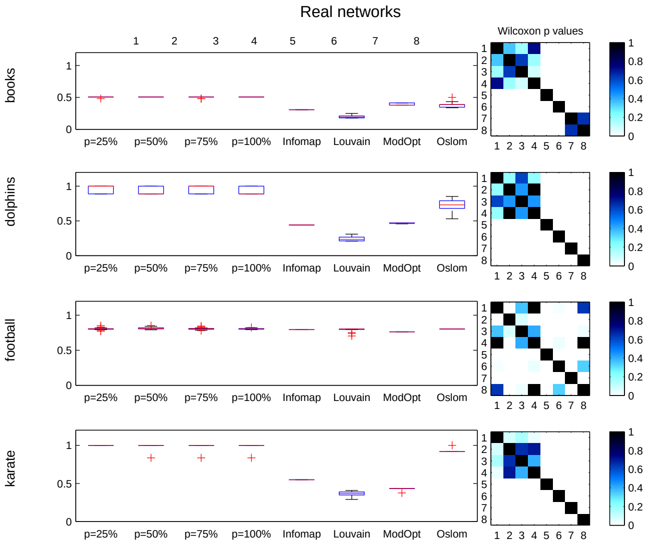
pMNEO


GN benchmark
Wilcoxon p values
z_{out}=6
z_{out}=7
z_{out}=8
p=25%
p=50%
p=75%
p=100%
Infomap
Louvain
ModOpt
Oslom

pMNEO


LFR benchmark, 1000 nodes, small
Wilcoxon p values
1 2 3 4 5 6 7 8
μ=0.1
μ=0.2
μ=0.3
μ=0.4
μ=0.5
p=25% p=50% p=75% p=100% Infomap Louvain ModOpt Oslom
0 0.5 1
0 0.2 0.4 0.6 0.8 1

*p*MNEO


Real networks
Wilcoxon p values
books
dolphins
football
karate
p=25%
p=50%
p=75%
p=100%
Infomap
Louvain
ModOpt
Oslom

*p*MNEO

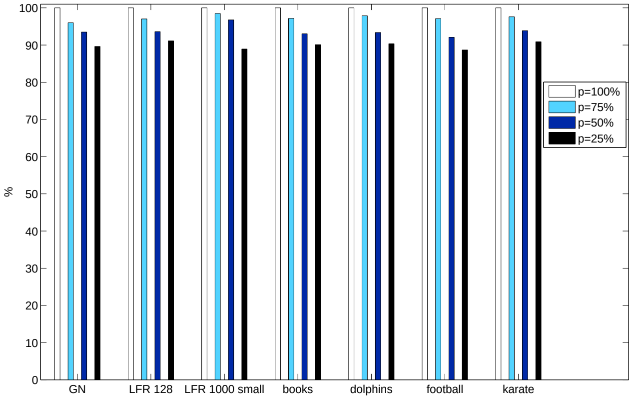


Figure: pMNEO duration in percents relative to $p = 100\%$

ϵ-Berge equilibria & the community detection problem

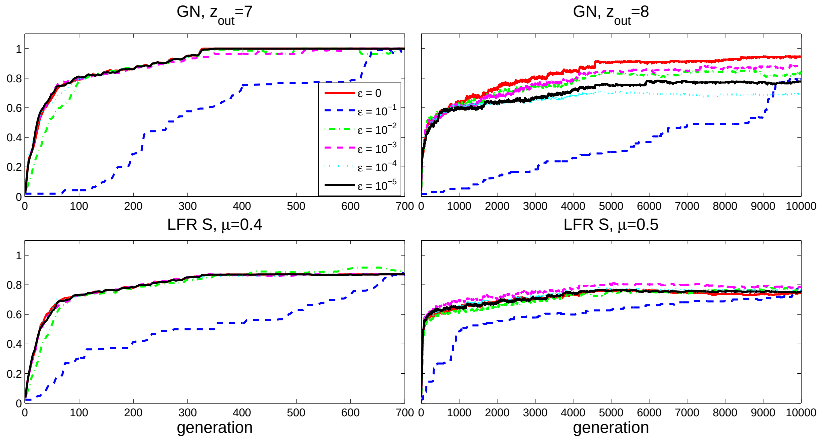


Figure: Evolution of NMI for different values of ϵ

ϵ-Berge equilibria & the community detection problem

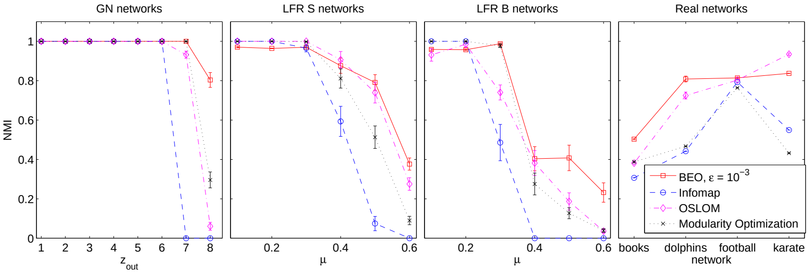


Figure: Berge equilibria - comparisons with other methods

Conclusions:

- (some) equilibria can be characterized by the use of "generative relations";
- generative relations can be embedded in evolutionary algorithms to compute corresponding equilibria;
- Extremal optimization is efficient for multi-player games;
- The community detection problem can be approached as a game;
- EO results are promising with both NE and BZ!

The work presented here was the result of the collaboration with:

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- Mihai Alexandru Suciu
- Tudor Dan Mihoc
- prof. D. Dumitrescu

Thank you for your attention!

Questions?

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