Equilibria Extremal Optimization

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1. Game theory

2. Nash Extremal Optimization

3. Application...
Solution concepts

Problem description:
- Game → players, strategies, payoffs;
- Multiobjective optimization → optimize multiple, conflicting objectives;

Solution concepts:
- Nash equilibrium (NE): no player can improve its payoff by unilateral deviation
- Berge equilibrium (BE): players maximize each others payoffs
- Pareto optimal solution/equilibrium: no objective can be increased without decreasing another objective
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Nash ascendancy relation

Compare two strategy profiles:

- compute $k(s, q) = \text{card}\{i \in N|u_i(q_i, s_{-i}) > u_i(s), q_i \neq s_i\}$
- $s$ Nash ascends $q$ if $k(s, q) < k(q, s)$
- $s$ non-dominated with respect to the Nash ascendancy relation: $\not\exists q \in S$ such that $q$ Nash ascends $s$.
- Nash non-dominated $= \text{NE}$. 
Reduce the number of payoff function evaluations:

- only a percent $p$ of players are tested:
- $I_p = \{i_1, i_2, ..., i_{np}\} \subset N$
- then $k_p(s, q, I_p) = \text{card}\{i \in I_p | u_i(s) < u_i(q_i, s-i), s_i \neq q_i\}$.
- $s \in S$ $p-$Nash ascends $q \in S$ with respect to $I_p$ if $k_p(s, q, I_p) < k_p(q, s, I_p)$.
- $s \in S$ $p$-non-dominated $\nexists q \in S, \nexists I_p \subset N$ such that $q$ $p$-ascends $s$ with respect to $I_p$.
- $p$-non-dominated $= \text{Nash equilibria}$
Berge generative relation

Definition (Berge-Zhukovskii)
A strategy profile $s^* \in S$ is a Berge-Zhukovskii equilibrium if the inequality
\[ u_i(s^*) \geq u_i(s^*_i, s_{-i}) \]
holds for each player $i = 1, \ldots, n$, and all $s_{-i} \in S_{-i}$.

Quality measures:
\begin{align*}
    b(s, q) &= \text{card}\{i \in N, u_i(s) < u_i(s_i, q_{-i}), s_{-i} \neq q_{-i}\}, \\
    b_\epsilon(s, q) &= \text{card}\{i \in N, u_i(s) < u_i(s_i, q_{-i}) + \epsilon, s_{-i} \neq q_{-i}\},
\end{align*}

The same mechanism $\rightarrow$ generative relation
Evolutionary detection - Nash equilibria

Game:

- Payoffs:

  \[ u_1(y_1, y_2) = y_1 \]
  \[ u_2(y_1, y_2) = (0.5 - y_1)y_2 \]

- \( y_1 \in [0, 0.5], \ y_2 \in [0, 1] \).
- NE: \((0.5, \lambda), \ \lambda \in [0, 1]\).

NSGA-II results
Example

\[ u_1(s_1, s_2) = -s_1^2 - s_1 + s_2, \]
\[ u_2(s_1, s_2) = 2s_1^2 + 3s_1 - s_2^2 - 3s_2, \]
\[ s_i \in [-2, 1], i = 1, 2. \]

Berge-Zhukovskii equilibrium: \((1, 1)\).

Payoffs \((-1, 1)\).

\[ \epsilon \in \{0, 0.1, 0.2, 0.5, 0.9\}. \]
Cournot oligopoly

- \( q_i, \ i = 1, \ldots, N \) - quantities/ N companies
- market clearing price \( P(Q) = a - Q \), where \( Q \) is the aggregate quantity on the market.
- total cost for the company \( i \) of producing quantity \( q_i \):
  \[ C(q_i) = cq_i. \]
- payoff for the company \( i \) is its profit:
  \[ u_i(q_1, q_2, \ldots, q_N) = q_iP(Q) - C(q_i) \]
- if \( Q = \sum_{i=1}^{N} q_i \), the Cournot oligopoly has one Nash equilibrium
  \[ q_i = \frac{a - c}{N + 1}, \forall i \in \{1, \ldots, N\}. \]
Multi-player games - probabilistic Nash ascendancy

Differential evolution

CrDE

number of players/payoff functions - q parameter

distance to Nash equilibrium
Algorithm 1 Nash Extremal Optimization procedure

1: Initialize configuration $s = (s_1, ..., s_n)$ at will; set $s_{\text{best}} := s$;
2: repeat
3: For the 'current' configuration $s$ evaluate $u_i$ for each player $i$;
4: find $j$ satisfying $u_j \leq u_i$ for all $i, i \neq j$, i.e., $j$ has the ”worst payoff”;
5: change $s_j$ randomly;
6: if (s Nash ascends $s_{\text{best}}$) then
7: set $s_{\text{best}} := s$;
8: end if
9: until a termination condition is fulfilled;
10: Return $s_{\text{best}}$.  

Multi-player games

Nash Extremal Optimization - Cournot

The diagram illustrates the distance to Nash Equilibrium (NE) for different game scenarios with varying numbers of players. The x-axis represents different game scenarios (NEO-10, CrDE-10, NEO-50, CrDE-50, NEO-100, CrDE-100, NEO-250), and the y-axis represents the distance to the Nash Equilibrium on a logarithmic scale. The data points and error bars indicate the variability and central trends in the optimization outcomes for each scenario.
Multi-player games

Nash Extremal Optimization - Cournot

250->1000 players

distance to NE

method and number of players

NEO-250  CrDE-250  NEO-500  CrDE-500  NEO-750  CrDE-750  NEO-1000  CrDE-1000
Community structure detection in social networks

Community structure - definitions
The game

Game $\Gamma = (N, S, U)$:

- $N = \{1, \ldots, n\}$, the set of players: network nodes;
- $S = S_1 \times S_2 \times \ldots \times S_n$, the set of strategy profiles of the game; $s \in S$, $s = (c_1, \ldots, c_n)$
- $U = \{u_i\}_{i=1}^n$, the payoff functions; $u_i : S \rightarrow \mathbb{R}$ represents the payoff of player $i$, $i = 1, n$.

$$u_i(s) = f(C_i) - f(C_i \setminus \{i\}),$$

$$f(C) = \frac{k_{in}(C)}{(k_{in}(C) + k_{out}(C))^{\alpha}},$$

- $k_{in}(C)$ - double of the number of internal links in the community;
- $k_{out}(C)$ - number of external links of the community;
- $\alpha$ - a parameter controlling the size of the community.
Randomly initialize all individuals in $P$ and $A$; Evaluate all individuals in $P$ and $A$; 

for $NrGen = 0$ to $MaxGen$ do 
  Set $m_{ngen} = \max \left\{ 1, \left[ \frac{1}{10} \cdot N \cdot (N - 2)^{-\frac{NrGen}{MaxGen}} \right] \right\}$
  Run a NEO iteration for each pair $(P_i, A_i)$;
  if $\Lambda$ iterations were performed on the original network then
    Mix network with probability $\rho$;
    Randomly re-initialize population $A$;
    Evaluate all individuals in $P$ and $A$;
  end if
  if Network is mixed and $\lambda$ iterations were performed then
    Restore network to original structure;
    Randomly re-initialize population $A$;
    Evaluate all individuals in $P$ and $A$;
  end if
end for

Output: individual from $A$ having the best modularity;
Results - MNEO

Figure: Comparisons with other methods - mean NMI (with error bars) values obtained in 30 runs (30 instances for each GN and LFR networks and 30 independent runs for the real-world networks).
Figure: GN6-8; boxplots of NMI values for all methods considered (left). On the right, the color matrix represents Wilcoxon p values for each pair of methods considered, numbered in the order they appear in the boxplot. A white square indicates a statistical difference between results.

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LFR benchmark, 1000 nodes, small

Wilcoxon p values

µ = 0.1

µ = 0.2

µ = 0.3

µ = 0.4

µ = 0.5

Figure: LFR 1000 nodes, small. Boxplots of NMI values for all methods considered (left). On the right, the color matrix represents Wilcoxon p values for each pair of methods considered, numbered in the order they appear in the boxplot. A white square indicates a statistical difference between results.

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Real networks

Wilcoxon p values

Figure: real networks; boxplots of NMI values for all methods considered (left). On the right, the color matrix represents Wilcoxon p values for each pair of methods considered, numbered in the order they appear in the boxplot. A white square indicates a statistical difference between results.
Figure: pMNEO duration in percents relative to $p = 100\%$
\( \varepsilon \)-Berge equilibria & the community detection problem

**Figure:** Evolution of NMI for different values of \( \varepsilon \)
$\epsilon$-Berge equilibria & the community detection problem

![Figure: Berge equilibria - comparisons with other methods](image-url)
Conclusions:

- (some) equilibria can be characterized by the use of "generative relations";
- Generative relations can be embedded in evolutionary algorithms to compute corresponding equilibria;
- Extremal optimization is efficient for multi-player games;
- The community detection problem can be approached as a game;
- EO results are promising with both NE and BZ!
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Thank you for your attention!

Questions?
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Questions?