A Generative Relation for the Characterization of Nash Equilibria

an extension to mixed form equilibria

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Abstract

A generative relation for the characterization of Nash equilibria in mixed form is presented. Two strategy profiles are compared by counting the number of players that would increase their payoffs by unilaterally switching from one strategy profile to the other; the one with the smallest number of such players is considered "better" with respect to the Nash equilibrium. It is shown that all strategy profiles that are non-dominated with respect to this relation are also Nash equilibria of the game. Moreover, this relation can be embedded in optimization heuristics in order to direct their search towards Nash equilibria of a game.

Notations

A finite strategic game is defined by $\Gamma = ((N, S_i, u_i), i = 1, n)$ where:

- N represents the set of players, $N = \{1, ..., n\}$, n is the number of players;
- for each player $i \in N$, S_i represents the set of m_i actions available to him, $S_i = \{s_{i1}, s_{i2}, ..., s_{im_i}\}$; $S = S_1 \times S_2 \times ... \times S_N$ is the set of all possible situations of the game; an element $s \in S$ is a *pure* strategy profile of the game;
- for each player $i \in N$, $u_i : S \to \mathbb{R}$ represents the payoff function.

The following notations are based on [2]:

Let $\mathcal{P}_i$ be the set of real valued functions on S_i. The notation $p_{ij} = p_i(s_{ij})$ is used for elements $p_i \in \mathcal{P}_i$. Let $\mathcal{P} = \times_{i=1,...,N}\mathcal{P}_i$ and $m = \sum_{i=1}^n m_i$. Then $\mathcal{P}$ is isomorphic to $\mathbb{R}^m$. We denote elements in $\mathcal{P}$ by $P = (P_1, P_2, ..., P_N)$ where $P_i = (p_{i1}, p_{i2}, ..., p_{im_i})$. If $P \in \mathcal{P}$ and $P_i' \in \mathcal{P}_i$ then (P_i', P_{-i}) stands for the element $Q \in \mathcal{P}$ that satisfies $Q_i = P_i'$ and $Q_j = P_j$ for $j \neq i$.

Let Δ_i be the set of probability measures on S_i. We define $\Delta = \times_{i=1,...,N}\Delta_i$. Elements $p_i \in \Delta_i$ are real valued functions on S_i: $p_i : S_i \to \mathbb{R}$ and it holds that $\sum_{s_{ij} \in S_i} p_i(s_{ij}) = 1,\ p_i(s_{ij}) \geq 0,\ \forall s_{ij} \in S_i$.

We use the abusive notation S_{ij} to denote the strategy $P_i \in \Delta_i$ with $p_{ij} = 1$. Hence, the notation (S_{ij}, P_{-i}) represents the strategy where player i adopts the pure strategy S_{ij} and all other players adopt their components of P. The payoff function u_i is extended to have domain $\mathbb{R}^m$ by the rule $u_i(P) = \sum_{s \in S} P(s)u_i(s)$, where $P(s) = \prod_{i=1}^N P_i(s_i)$.

A strategy profile $P^* = (P_1^*, P_2^*, ..., P_N^*) \in \Delta$ is a *Nash equilibrium (NE)* if for all $i \in \{1, ..., N\}$ and all $P_i \in \Delta_i$, we have

$$u_i(P_i, P_{-i}^*) \leq u_i(P^*).$$

Nash ascendancy

The Nash ascendancy concept was introduced with the purpose to compare two strategy profiles [1] during the search of an evolutionary algorithm in order to compute NEs of a game.

Consider two strategy profiles P^* and P from Δ. Then $k : \Delta \times \Delta \to N$ associates the pair (P^*, P) the cardinality of the set

$$k(P^*, P) = card\{i \in \{1, ..., n\} | u_i(P_i, P_{-i}^*) > u_i(P^*), P_i \neq P_i^*\}.$$

This set is composed by the players i that would benefit if - given the strategy profile P^* - would change their strategy from P_i^* to P_i.

It is obvious that for any $P^*, P \in S$, we have

$$0 \leq k(P^*, P) \leq n.$$

Definition 1. *Let $P, Q \in \Delta$. We say the strategy profile P Nash ascends Q and we write $P \prec Q$ if the inequality*

$$k(P, Q) < k(Q, P),$$

holds.

Remark 2. *Two strategy profiles $P, Q \in \Delta$ can have the following relation:*

1. either P Nash ascends Q,

2. either Q Nash ascends P,

3. if $k(P, Q) = k(Q, P)$ then P and Q are indifferent

Definition 3. *A strategy profile $P^* \in S$ is called non-dominated with respect to the Nash ascendancy relation (NNS) if*

$$\nexists Q \in \Delta, Q \neq P^* \text{such that } Q \prec P^*.$$

Definition 4. *The set of all Nash nondominated strategy profiles with respect to the Nash ascendancy relationo is the set containing all nondominated strategies i.e.*

$$NND = \{s \in S | s \text{ Nash non-dominated with respect to the Nash ascendancy relation}\}$$

Proposition 5. *A strategy profile $P^* \in \Delta$ is a NE iff the equality*

$$k(P^*, Q) = 0, \forall Q \in \Delta,$$

holds.

Proof. Let $P^* \in \Delta$ be a NE. Suppose there exists $Q \in \Delta$ such that $k(P^*, Q) = w$, $w \in \{1, ..., n\}$. Therefore there exists $i \in \{1, ..., n\}$ such that $u_i(Q_i, P_{-i}^*) > u_i(P^*)$ and $Q_i \neq P_i^*$, which contradicts the definition of NE.

For the second implication, let $P^* \in \Delta$ such that $\forall Q \in \Delta, k(P^*, Q) = 0$. This means that for all $i \in \{1, ..., n\}$ and for any $Q_i \in \mathcal{P}_i$ we have $u_i(Q_i, P_{-i}^*) \leq u_i(P^*)$. It follows that P^* is a NE. □

Proposition 6. *All NE are Nash nondominated solutions (NND) i.e.*

$$NE \subseteq NND.$$

Proof. Let $P^* \in \Delta$ be a NE. Suppose that there exists a strategy profile $P \in \Delta$ such that $P \prec P^*$. It follows that $k(P, P^*) < k(P^*, P)$. But $k(P^*, P) = 0$, therefore we must have $k(P, P^*) < 0$ which is not possible since $k(P, P^*)$ denotes the cardinality of a set. □

Proposition 7. *All Nash nondominated solutions are NE, i.e.*

$$NND \subseteq NE.$$

Proof. Let P^* be a nondominated strategy profile. Suppose P^* is not NE. Therefore there must exist (at least) one $i \in \{1, ..., n\}$ and a strategy $P_i \in \mathcal{P}_i$ such that

$$u_i(P_i, P_{-i}^*) > u_i(P^*),$$

holds. Let's denote by $Q = (P_i, P_{-i}^*)$. It means that $k(P^*, Q) = 1$. But $k(Q, P^*) = 0$. Therefore $k(Q, P^*) < k(P^*, Q)$ which means that $Q \prec P^*$ thus the hypothesis that P^* is nondominated is contradicted. □

Using propositions 6 and 7 it is obvious that the next result holds:

Proposition 8. *The following relation holds:*

$$NE = NND,$$

i.e. all NE are also Nash nondominated and also all Nash nondominated strategies are NE.

Examples

The Nash ascendancy relation defined above can be considered as a generative relation for Nash equilibria in mixed form. This means that this relation can be embedded into search heuristics that, by successively selecting Nash-nondominated strategies from a population, compute mixed Nash equilibria of a game. An example of results obtained with such a method for the following four games from the GAMBIT distribution [3] are presented below.

Game	No. of players	No. of strategies	Type of NE
G1	3	2,2,2	totally mixed
G2	3	3,3,3	mixed
O'Neill	2	4,4	totally mixed
Poker	2	4,2	totally mixed

Descriptive statistics (over 30 independent runs) of numerical results obtained by the Direct Evolutionary Search algorithm that uses the Nash ascendancy relation:

G1			
Avg. dist:	4.04061E-11	St dev:	2.25E-17
Avg. no. gen.:	11,941	St dev:	13424.39
Avg. eval.:	15,541,134	St dev:	17317631
G2			
Avg. dist:	4.04061E-11	St dev:	6.98E-18
Avg. no. gen.:	931	St dev:	324.70
Avg. eval.:	1,076,041	St dev:	374858.2
Oneill			
Avg. dist:	2.79146E-05	St dev:	0.0001
Avg. no. gen.:	8,217.63	St dev:	26628.89
Avg. eval.:	6,556,652.7	St dev:	21598601
Poker			
Avg. dist:	3.70074E-18	St dev:	1.99E-17
Avg. no. gen.:	794.33	St dev:	240.73
Avg. eval.:	419,827.93	St dev:	126713.48

References

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[3] Richard D. McKelvey, Andrew M. McLennan, and Theodore L. Turocy. Gambit: Software tools for game theory. Technical report, 2010.